Prospective Secondary Teachers' Subject Matter Knowledge and Pedagogical Content Knowledge of the Concept of Function

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PROSPECTIVE SECONDARY TEACHERS’ SUBJECT MATTER KNOWLEDGE AND PEDAGOGICAL CONTENT KNOWLEDGE OF THE CONCEPT OF FUNCTION

By

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This dissertation is dedicated to my family members:

my older brother, Selcuk,
for always encouraging me every step of the way and believing in me to pursue my dreams and
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ABSTRACT

The current study examined two prospective secondary mathematics teachers’ subject matter knowledge and pedagogical content knowledge of the concept of functions as they completed a function questionnaire, card sorting activity, analysis of two lesson plans and their video recorded teaching, preparation and analysis of lesson plan on exponential functions, and video teaching episode of the lesson plan on exponential functions during six weeks of a teaching sequence. The task-based activities that the prospective secondary mathematics teachers were involved in during this study were analyzed to gain insight into their knowledge of the concept of functions. Even’s (1989) framework was used to examine the participants’ subject matter knowledge and pedagogical content knowledge of the concept of functions. The model of Wilson, Shulman and Richert (1987) was used to organize the tasks and investigate whether or not their pedagogical content knowledge improved as a result of doing these tasks. The study also examined the relationship between the prospective secondary mathematics teachers’ subject matter knowledge and pedagogical content knowledge of the concept of functions.

The results of the study revealed that the participants mentioned the nature of univalence property but did not know the arbitrary nature of functions. The study also revealed excessive use of the vertical line test without explaining what it meant to fail the test, why it worked or why it was or was not a function. The participants’ understanding of functions among different representations was very weak and not connected. Participants’ weak subject matter knowledge as well as weak pedagogical content knowledge prevented them from taking students misconceptions, sources of incorrect solutions, into account. The participants did gain some experience in preparing and teaching a lesson. However, they did not show a big improvement during the lesson plan analysis. In their teaching and the evaluation of their teaching, they pointed out important issues: strong and weak aspects of lesson plans, things to change in their lesson, and weakness and strength of their teaching.
CHAPTER 1

INTRODUCTION

The concept of function is one of the most essential and fundamental concepts in mathematics (Dreyfus & Eisenberg, 1983; Even & Tirosh, 1995) and appears in all areas of mathematics. The significance of functions is clearly seen by their large network of relations to other concepts as well as different branches of mathematics such as algebraic operations on numbers and transformations on points in the plane or in the space (Hansson, 2004; Dreyfus & Eisenberg, 1984; Even, 1989). One example of function is the relationship between the speed of a car and distance traveled. In the school curriculum, function is central and usually introduced with a variety of representations such as arrow diagrams, tables, algebraic formulas, and graphs (Dreyfus & Eisenberg, 1984).

While central to the mathematics curriculum, there is much to be learned about students and teachers’ understanding of functions. The dynamics of a mathematics classroom is influenced by several elements: such as the students’ understanding of the mathematics, the teachers’ understanding of the mathematics, in teaching and in learning and management of the learning environment. The interaction of all such variables affects what goes on during the activities of teaching and learning.

Students’ Understanding of Functions

Research studies have been conducted to examine high school students’ understanding of functions (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Stump, 2001; Zazkis, Liljedahl, & Gadowsky, 2003). The results of these studies revealed that students’ understanding of the concept of function is a complex process and difficult to understand (e.g., Bell & Janvier, 1981; Leinhardt, Zaslavsky, & Stein, 1990; Vinner and Dreyfus, 1989). The concept of function is complex due to several factors (Leinhardt, Zaslavsky, & Stein, 1990): (1) it is related with other complex mathematical concepts, (2) it is interactive with various sub-concepts, and (3) it is denoted by varying representations (Dreyfus & Eisenberg, 1982). Because of the complex structure of the concept of function, students rarely develop an adequate understanding of it (Dreyfus and Eisenberg, 1983). Understanding of the notion of a function involves the mastery of many different aspects of the concept (e.g., Dreyfus & Eisenberg, 1984, Even, 1989; Stump,
Hansson (2004) suggests that students develop a good understanding of the function concept when they comprehend the knowledge of the concept of function and its relations to other concepts, and are able to use functions in different contexts. Hansson argues that learning and being able to use functions in different contexts requires a longer period of time of study.

When the students are asked whether or not the graphs were functions (e.g., Markovits, Eylon & Bruckheimer, 1986; Vinner & Dreyfus, 1989), they failed to recognize piece-wise functions and constant functions as a function. Due to their poor concept image of functions, students think that graphs of functions should be familiar with their mental image and definition about the concept of function. Students’ concept image and concept definition are based on the functions that they learned rather than modern definition of the concept of function. Students typically thought that the graphs of functions should be represented by a formula (Tall, 1992; Vinner, 1983; Vinner & Dreyfus, 1989).

Stump (2001) examined twenty-two high school pre-calculus students’ understanding of slope and found that many students had difficulties in interpreting slope as a measure of rate of change. The researcher suggests that a focus of the instruction be to help students create connections among rates and other variables, and graphical representations of these relationships. The teachers create opportunities for the students to investigate both physical and functional situations when they use real-world examples to illustrate the concept of slope. In their study, Zazkis, Liljedahl, and Gadowsky (2003) investigated ten secondary students’ understanding of horizontal shift of a parabola. The result of the study revealed that the students relied on memorized rules. The students did not use a point-wise approach due to lack of experience with drawing graphs manually.

**Teachers’ Understanding of Functions**

A teacher’s subject matter knowledge influences his/her way of teaching to the students. The teacher who has strong mathematical knowledge is more competent to help his/her students attain a meaningful understanding of the subject matter (Even, 1990). The teacher asks questions, stimulates discussions, and suggests different points of view to the students. These activities and decisions require teachers to have adequate subject matter knowledge as well as pedagogical content knowledge (Even, 1989). When prospective teachers have misconceptions or limited content knowledge, they may pass on these misconceptions to their students. They may fail to challenge them (Ball & McDiarmid, 1990). Their conceptions might limit their
ability to present subject matter in appropriate ways, give helpful explanations and conduct discussions (Even & Tirosh, 1995).

Subject matter knowledge of mathematics includes both knowledge of mathematics and knowledge about mathematics (Ball, 1990c). Knowledge of mathematics includes “understanding of principles and meaning of underlying mathematical procedures” (Ball, 1990c, p. 6). Knowledge about mathematics involves

Understanding of the nature of knowledge in the discipline: where it comes from, how it changes, and how truth is established; the relative centrality of different ideas as well as what it is conventional or socially agreed upon in mathematics versus what is necessary or logical (Ball, 1990c, p. 6).

In addition, a teacher should be able to explain why a particular statement is important to be demonstrated and why it is worth knowing both within the discipline and without (Shulman, 2004). The teacher must understand and use different ways of organizing the discipline and be able to recognize alternative ways of representing the subject matter.

Even (1990) found that prospective teachers’ knowledge of functions was weak and fragile. Prospective teachers did not have a modern concept of function. Their conceptions of functions were similar to the high school students described by researchers (e.g., Tall, 1992): the prospective teachers indicated that graphs of functions should be familiar with their mental image and definition about the concept of function.

Research has also shown that it is difficult to change prospective teachers’ perspectives about teaching and learning of functions (e.g., Sanchez & Llinares, 2003). Wilson (1994) examined the prospective teacher’s understanding of the concept of function. Even though he used extensive materials that included graphical, algebraic representations of functions, non-functions, and real life applications of functions, the prospective secondary teacher’s perspective of mathematics and mathematics teaching was still relatively narrow at the end of the study. Even (1993) investigated prospective teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function and found that many of the prospective secondary teachers did not have the appropriate concept image and concept definition of a function. Many of the prospective teachers did not accept unfamiliar functions. They thought that familiar functions were always represented as equations or formulas, or graphs. A majority of the prospective teachers in the study Zazkis, Liljedahl, and Gadowsky (2003) relied on memorized rules (i.e., the law of opposites) when they were asked to explain how they sketched the graph of
$y = (x - 3)^2$. The prospective teachers’ difficulty with explaining behavior of the parabola highlighted that they pay attention to memorization rather than reasoning.

**Pedagogical content knowledge**

Pedagogical content knowledge consists of an understanding of what makes the learning of a specific topic easy or difficult in mathematics and what the students bring with them in terms of learning most commonly taught topics and lessons. Pedagogical content knowledge also includes most useful representations, examples, explanations, and the ways of representing the most frequently taught topics and lessons comprehensible to students (Shulman, 2004).

Ball (1990c) stated that subject matter knowledge should be a central focus of teacher education in order to teach mathematics effectively. I believe that both subject matter knowledge and pedagogical content knowledge should be a central focus in teacher education. In order to teach effectively, teachers’ understanding of the concept of function must rely on solid knowledge of subject matter. Moreover, under different circumstances, the teacher should be able to recognize the pedagogical grounds for selecting alternative ways of representing the subject matter. Accordingly, the quality of teaching must rest on features of pedagogical reasoning that can be explained through pedagogical actions (Shulman, 2004). The prospective teachers develop their pedagogical content knowledge as they plan to teach as well as during actual teaching (Wilson, Shulman & Richert, 1987). However, effective teaching requires understanding what students know and need to learn as well as challenging and supporting them to learn mathematics (National Council of Teachers of Mathematics [NCTM], 2000).

Research studies focused on prospective teachers’ pedagogical content knowledge of the concept of function indicated that prospective teachers had difficulties working with different representations of functions (Even, 1990, 1998). Even (1993) found that the prospective teachers failed to recognize unfamiliar functions as a function. Even and Tirosh (1995) stated that the prospective teachers did not have a sufficient understanding of the role of univalence and relied on the vertical line test to present the concept of function to the students. Even and Tirosh (1995) stated that not knowing why the rule works affected the prospective teachers’ pedagogical content-specific choices. Many of the prospective teachers had trouble explaining because their pedagogical content knowledge for teaching was not comprehensive and articulated. Sanchez and Llinares (2003) examined the influence of prospective teachers’ subject matter knowledge and perspectives of mathematics about teaching and learning on their hypothetical
presentation of subject matter for teaching in the context of functions. They found that all four prospective teachers described teaching as telling, learning and remembering. All four prospective teachers looked at mathematics as a set of systematic procedures for solving problems. There was a slight change on the influence of their perspectives of mathematics and teaching and learning on prospective secondary teachers’ organization of the subject matter for teaching at end of the study.

I believe it is important for mathematics teachers to have a well-developed conceptual knowledge of functions, and to be successful in dealing with the concept of function in their practices. Because mathematics teachers are responsible for guiding students to extend their knowledge of functions, applications and representations, they should develop the concept’s significance in mathematics and relations to other concepts. In order for prospective teachers to be successful in dealing with the concept of function in teaching practices, they need to have strong subject matter knowledge and pedagogical content knowledge of the concept of function.

Research Questions

The following research questions was addressed in this study:

1. What is the nature of subject matter knowledge of prospective secondary mathematics teachers for teaching the concept of function?
   a) How do prospective secondary mathematics teachers classify relations into functions and non-functions?
   b) What are the prospective secondary mathematics teachers understanding of the relationships among different types of functions?
   c) How do the prospective secondary mathematics teachers translate from one representation to another?

2. What is the nature of pedagogical content knowledge of prospective secondary mathematics teachers for teaching the concept of function?

3. What is the nature of relationships between prospective secondary mathematics teachers’ subject matter knowledge and pedagogical content knowledge for teaching the concept of function?

Definition of key terms

In this research study, it is important to define the key terms to have a clear understanding of what this study focused on. Definitions of these key terms are as follow:
I. Subject matter knowledge: Subject matter knowledge includes understanding of the basic concepts and principles of the concept of function in variety of ways as well as the relationships among them. Prospective teachers must understand and be able to explain why a particular proposition such as univalence and arbitrariness is worth knowing and how it is related to other concepts both within the discipline and without, both in theory and practice. The prospective teachers should be able to understand the concept of function in different representations such as illustrations, examples and explanations and be able to interpret and form connections among and between them in the different divisions of mathematics, other disciplines or everyday life. With use of different representations, the prospective teachers acquire representational repertoire that shows a good understanding of the concept of function in the discipline of mathematics. They must have a deeper understanding of general and more complicated knowledge and be able to represent essential principles and properties and have easy access to specific examples. The prospective teachers must learn procedural (i.e., formal language of mathematics and algorithms for completing mathematical tasks) and conceptual knowledge (i.e., recognizing and generating relationships between units of knowledge) and the relationship between them, and be able to understand and use inductive and deductive reasoning in mathematics.

II. Pedagogical content knowledge: My definition of pedagogical content knowledge includes three large clusters of knowledge. The first two clusters are based on Shulman’s (1986) definition and the last cluster is from Graber’s (1999) study. (1) It consists of the most useful forms of representations and ideas, such as illustrations, examples and explanations as well as the ways of representing and formulating the concept of function to make it comprehensible to others as well as an understanding of what makes the learning of those most frequently taught topics and lessons easy or difficult. (2) Pedagogical content knowledge also involves knowing and recognizing what makes the learning of specific topics in concept of function easy or difficult and the conceptions and preconceptions that students bring with them to the learning of those most frequently taught topics in the concept of function. (3) It includes valuing students’ current understanding to make instructional decisions and being skillful in differentiating what the students comprehend and what the students can perform (Graber, 1999) as well as providing the grounds for choices and actions in one’s pedagogical decisions (Shulman, 2004). One’s understanding and knowledge of concept must be related to judgment and action and to appropriate uses of understanding in his/her pedagogical decisions (Shulman, 2004).
conception, and pedagogical reasoning and action can be examined through the activities of comprehension, transformation, instruction, evaluation, and reflection (Wilson, Shulman & Richert, 1987).

**III. Concept of function:** The definition of function simply uses the idea of univalence that for each element in the domain there is exactly one element in the codomain, with no other required properties of the correspondence. Arbitrariness and univalence are the essential features of the concept of function (Freudenthal, 1983 cited in Even (1990)). Arbitrariness of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. The arbitrary nature of the two sets means those functions do not have to be described on any specific set of objects. Arbitrariness is closely interrelated to an analytical judgment when an instance belongs to a concept family (Even, 1990).

**IV. Nature of knowledge:** The essential characteristics and qualities of prospective teacher that determines his/her characteristic actions and reactions.

**Significance**

Informed by the current research in the field, the proposed study expanded the set of studies in teaching and learning about functions. Thus, there were three primary reasons why I wanted to conduct a study to examine prospective secondary teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function.

First, research studies have shown that it is difficult for high school students to develop a good understanding of the concept of function concept due to its complex relation to other mathematical concepts and the different representations of the concept (e.g., Bell & Janvier, 1981; Leinhardt, Zaslavsky, & Stein, 1990; Vinner & Dreyfus, 1989). I believe that this study about prospective teachers’ subject matter knowledge and pedagogical content knowledge would help the mathematics education community better understand the factors behind students’ difficulties. The prospective teachers learn the concept of function extensively in high school. They bring what they learn in high school to formal teacher education. Therefore, it was very important to look at their difficulties and deficiencies.

Second, research studies have revealed that prospective teachers do not have adequate subject matter knowledge to teach mathematics (e.g., Ball, 1990a; Even 1990, 1993, 1998). Prospective teachers’ subject matter knowledge of the concept of function is weak (Even, 1990). For example, they are unable to identify functions and non-functions and to represent different
representations of functions (e.g., Even, 1993; Sanchez & Llinares, 2003). Moreover, prospective teachers’ pedagogical content knowledge of the concept of function is not comprehensive and articulated (e.g., Even & Tirosh, 1995; Sanchez & Llinares, 2003). This study added detailed information about prospective teachers’ deficiencies in their subject matter knowledge and pedagogical content knowledge as well as helped mathematics educators better determine what to focus on in teacher education practices.

Third, research studies (e.g., Even, 1990; Sanchez & Llinares, 2003) have investigated prospective secondary teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function using questionnaires. It is surprising to see that the focus of these research studies do not include examining prospective secondary teachers’ classroom instruction. This study went beyond using questionnaires and conducting interviews by including an examination of teaching. I believe that it was necessary to include the teaching aspect in order to better understand the relationship between the subject matter knowledge and pedagogical content knowledge of the concept of function because observation of students in teaching situations enabled me to see what they knew and how they used it in the classroom.
CHAPTER 2

REVIEW OF LITERATURE

Shulman (1986, 2004) distinguished and put great emphasis on three categories of content knowledge: (1) subject matter knowledge, (2) pedagogical content knowledge, and (3) curricular knowledge. The focus of the research questions is on the first two categories; and hence, chapter 2 will focus on the two categories of content knowledge: subject matter knowledge and pedagogical content knowledge.

Subject Matter Knowledge

Subject matter knowledge has been defined by different researchers. Wilson, Shulman and Richert (1987) posited that subject matter knowledge involves both substantive and syntactic structures of the discipline. The substantive structures involve the ideas, facts, and concepts of the field, as well as the relationships among those ideas, facts, and concepts (Wilson, Shulman & Richert, 1987). The substantive structures provide a variety of ways for teachers to incorporate and organize the basic concepts and principles of the discipline (Schwab, 1978, cited in Shulman, 2004).

On the other hand, the syntactic structure of a discipline provides us with a set of ways for teachers to establish truth and falsehood, validity or invalidity (Wilson, Shulman & Richert, 1987). The syntactic structures include “knowledge of the ways in which the discipline creates and evaluates new knowledge” (Wilson, Shulman & Richert, 1987, p. 118). A syntax is like a grammar. It is the set of rules for determining what is legitimate to say in a disciplinary domain and what “breaks” the rules (Wilson, Shulman & Richert, 1987).

Shulman (2004) stated that content knowledge refers to “the amount and organization of knowledge per se in the mind of the teacher” (p. 201). However, this does not mean that teachers should only be able to define the accepted truths of the domain to students. They must also be capable of explaining why a particular statement is necessary to consider or be demonstrated, why it is worth knowing, and how it relates to other statements, both within the discipline and without, both in theory and practice (Shulman, 2004). A teacher must understand that there are a variety of ways of organizing the discipline. The teacher should recognize
alternative forms of organization and the pedagogical grounds for selecting one under different circumstances. The teacher should also understand the syntax of the discipline. For Shulman (2004), to think properly about content knowledge requires more than knowledge of the facts or concepts of a domain. He stated,

The teacher need not only understand \textit{that} something is so; the teacher must further understand \textit{why} it is so, on what grounds its warrant can be asserted…The teacher needs to understand why a given topic is particularly central to a discipline whereas another may be somewhat peripheral (p. 202).

Subject Matter Knowledge for Teaching Mathematics

There has been an interest in defining and analyzing what subject matter knowledge is for teaching (Even, 1990). Ball (1990c) claimed that subject matter knowledge of mathematics includes both knowledge of mathematics and knowledge about mathematics. Understanding mathematics for teaching requires both knowledge of mathematics (i.e., understanding of principles and meaning of underlying mathematical procedures) and knowledge about mathematics (i.e., understanding of the nature of knowledge in the discipline: where it comes from, how it changes, and how truth is established; the relative centrality of different ideas as well as what it is conventional or socially agreed upon in mathematics versus what is necessary or logical, Ball 1990c, p.6). At this point, the teacher’s role is to help the students to attain understanding of the subject matter (Even, 1990). In order to help the students, “the teacher needs to have solid knowledge of the subject matter. A teacher, who has a solid mathematical knowledge for teaching is more capable of helping his/her students to achieve a meaningful understanding of the subject matter” (Even, 1990, p. 521).

Similarly, Even and Tirosh (1995) defined that the most basic level of subject matter is \textit{knowing that} and \textit{knowing why}. \textit{Knowing that} involves knowledge and rules, algorithms, procedures and concepts that are related to specific mathematical topics in the school curriculum. Even and Tirosh (1995) indicated that \textit{knowing that} is very important, because it includes a basis for adequate pedagogical content knowledge: for questions teachers ask and activities they suggest. Even and Tirosh (1995) stated that \textit{knowing why} includes “knowledge which pertains to the underlying meaning and understanding of why things are the way they are, enables better pedagogical decisions” (p. 9). \textit{Knowing why} also affects teacher’s decisions about the presentation of the subject matter. As it can be seen, Even and Tirosh’s definition of subject
matter knowledge is based on knowing that and knowing why. These concepts are the basis of making good pedagogical decisions for teaching.

Research Studies on Subject Matter Knowledge for Teaching Functions

Research studies examined prospective secondary teachers’ knowledge of the concept of function (e.g., Even & Tirosh, 1995; Sanchez & Llinares, 2003; Wilson, 1992, 1994) as well as college students’ misconceptions and difficulties (e.g., Dreyfus & Eisenberg, 1983; Vinner & Dreyfus, 1989).

In their study, Dreyfus and Eisenberg (1983) examined eighty-four college students’ perspective on the concept of function as well as on the mental image of function. Concept image is defined as the mental picture of this concept (the set of all pictures) that has ever been associated with the set of properties within the concept (in the person’s mind) (Vinner, 1983).

The researchers adopted this approach in order to obtain a better insight into their difficulties and intuitions on the basic notions. In their study, eighty-four college students from two Israeli universities participated in a questionnaire that included thirty-four questions, which focused on the concept of linearity, smoothness (differentiability) and periodicity that was internalized by college level students. The researchers examined whether or not the questions suggested linearity and to which extent linearity was clear to the college students. The difference was seen in the reactions of the college students for the algebraically stated problems that did not suggest linearity. The college students felt more comfortable with graphical exercises than algebraic ones. However, some of them thought that the graphical answers were not mathematical answers. The college students tended to use linearity and failed to satisfy functionality. They also thought algebraic data and graphical data as being independent of one another. The students did not even consider the unifying interrelationships between the various mechanical procedures.

Correspondingly, Vinner and Dreyfus (1989) compared the concept definition of 271 college students and 36 junior high school teachers and their images for the concept of mathematical functions. They designed questions to examine the function image of the college students and their definitions. Based on the level of mathematics courses required for their major, the college students were divided into four groups: low level, intermediate level, high level and mathematical level. Vinner and Dreyfus (1989) used both strong graphical and strong non-graphical aspects of the function concept. When the function concept did not include strong
graphical aspects, the image consisted of symbolic representations or formulas, as well as the set of all properties associated with the concept. Except for the mathematics majors and the teachers’ definitions, many of the definitions and even more of the images were primitive. The higher-level participants justified most of their correct answers better and had a broad perspective for their explanations than did the lower level respondents. However, the higher-level participants did give wrong answers and they showed similar behavior to that of the lower-level participants.

Dreyfus and Eisenberg (1982) stated that if one wants to teach the concept of function to a group similar to one of the groups in the study, it is important to know the starting point of its members. In their study the starting point of the researchers (Dreyfus & Eisenberg, 1982) was to divide college students based on the level of mathematics courses required for their major. In the study of Markovits, Eylon and Bruckheimer (1986), the starting point of a general understanding of the concept of function involving many aspects were the following (i.e., classify relations into functions and non-functions, giving examples of relations, which are functions, and non-functions, identifying preimages, images and (preimage, image) pairs for a given function, finding the image of a given preimage, identifying identical functions, translating from one representation to another, identify functions satisfying some given constraints).

Research Studies on Prospective Secondary Teachers’ Subject Matter Knowledge of Functions

Through her framework developed in 1989, Even (1990, 1993,1998) focused on investigating prospective secondary teachers’ knowledge and understanding of mathematical functions. Even’s (1990) research shows how one may approach the question of teachers’ knowledge about mathematical topics. It demonstrated the building of an analytic framework of subject matter knowledge for teaching in the case of the concept of function. Even (1990) stated that teachers’ subject matter knowledge about a specific mathematical topic is influenced by what they know across different domains of knowledge. Therefore, Even (1990) stated that analysis of teachers’ subject matter knowledge about a specific piece of mathematics should integrate seven aspects of knowledge: essential features, different representations, the strength of the concept, basic repertoire, alternative ways of approaching, different kinds of knowledge and understanding of function and mathematics, and analysis of students’ mistakes. The framework was used to analyze prospective secondary teachers’ subject matter knowledge of the
concept of function (Even, 1990). By using the findings and examples from the study of Even (1990), these seven bodies of knowledge are explained below.

**The Framework of Even (1989)**

*Essential feature*

Essential feature deals with the concept image (mental picture of concept) and essence of the concept. Vinner (1983) describes concept image as the mental picture of this concept (i.e., the set of all pictures that have ever been associated with the concept in person’s mind). Concept image may also be associated with the set of properties within the concept (in the person’s mind) (Vinner, 1983). Even (1990) argued that teachers should have a good correspondence between their understanding of a specific mathematical concept they teach and the ‘correct’ mathematical concept. Although there is no definite answer to the definition of a correct mathematical concept according to Even (1990), correct mathematical concept means that the teachers should be able to determine whether or not an instance belongs to a concept family by using analytical judgment, as opposed to a mere use of a prototypical judgment. The first type, analytical reasoning, is based on the concepts critical attributes. It means that an instance must have attributes in order to be a concept example. The second type, prototypical judgment, is based on applying a visual judgment to other instances or the judgment on the prototype’s self attributes. Although one develops concept image mainly from instances that he/she experiences, the concept image is not determined only by concept definition (Even, 1990). Teachers also should be able to correctly distinguish between concept examples and non-examples. If one has a limited and underdeveloped concept image, they may have to deal with unfamiliar situations where their students explore and raise questions. She emphasized that “teachers’ pedagogical decisions-questions they ask, activities they suggest—are based on their subject matter knowledge. Therefore, it is necessary that teachers are able to distinguish between concept examples and non-examples” (Even, 1990, p. 524).

Two essential features of the concept of function are arbitrariness and univalence (Freudenthal, 1983 cited in Even, 1990).

**Arbitrariness:** Arbitrariness of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. Arbitrariness does not have to be defined by any specific expression, follow some regularity, or be described by a graph with any particular shape. For instance, the function that describes the relationship between time and
temperature is an example of arbitrariness of functions. The arbitrary nature of the two sets means those functions do not have to be described on any specific set of objects. Specifically, the sets do not have to be sets of numbers. For instance, a rotation of the plane is an example of this type of functions, because it is described on points. “In addition to the arbitrariness of the relationship between the variables, themselves or the sets on which the function is defined were allowed to be any sets-arbitrary sets” (Even, 1990, p. 528). Even (1990) stated that arbitrariness is closely interrelated to an analytical judgment when an instance belongs to a concept family. In her study, Even (1990) found that the prospective secondary teacher had difficulty when he was asked to determine whether or not the unfamiliar graph was a graph of function. Even (1990) also stated that the prospective secondary teacher’s difficulty was caused by the inconsistency between his concept image of a function and the formal concept definition. His concept image of a function is determined by the functions he learned and not by the modern definition of a function that stresses the arbitrary nature of functions.

Univalence: Univalence is a function specific characteristic (Even, 1990). The univalence requirement is explicitly stated that for each element in the domain there is only one element (image) in the range. Univalence activity is commonly used for distinguishing between functions and non-functions in most texts on functions. The prospective secondary teachers should know that functions have to be univalent. In her study, Even (1990) found that even though most of the prospective secondary teachers knew about the univalence property of functions and its use for telling whether or not an unknown graph was a function, almost none of them explained why univalence was important and how functions came to be defined that way. In addition, many of the teachers were unable to explain, “What is it that you can do with functions that you cannot do with relations which are not functions” (Even, 1990, p. 531). Therefore, Even (1990) claimed “the prospective secondary teachers should have a good correspondence between their concept image of a function and the modern mathematical concept… They need to know why functions are defined this way” (Even, 1990, p. 530).

Different representations

Different representations mean that one understands the concept in different illustrations and be able to interpret and form connections among and between them. Even (1990) stated that teachers need to understand that different representations provide different insight and allow a better, deeper, more powerful and more complete understanding of a concept. For instance,
prospective secondary teachers had studied about and used quadratic functions since they were in high school; still, they viewed a quadratic expression only with graphical representation (Even, 1990). Many of the prospective secondary teachers saw the equation, \( ax^2 + bx + c = 0 \) would be same as graphical representation of the function \( f(x) = ax^2 + bx + c \). Although they were familiar with quadratic functions in both symbolic and graphic representation since high school, they were unable to make connections between the two (symbolic and graphic) representations. Many of the prospective secondary teachers’ lack of relationships and connectedness between the two representations prevented them from relating the given equation \( ax^2 + bx + c = 0 \), to a graphical representation of the function \( f(x) = ax^2 + bx + c \). Similarly, Eisenberg and Dreyfus (1986) found that very few prospective secondary teachers used graphical solutions for solving inequalities. Therefore, it is very important for prospective secondary teachers to have different representations of functions for teaching.

**Alternative ways of approaching**

*Alternative ways of approaching* refers to different uses of the concept in the different divisions of mathematics, other disciplines or everyday life. These alternative ways are different from one another and none of them are suitable for all situations. In some cases, more than one approach can be used, because some approaches are more appropriate than others. Therefore, teachers should acquire a representational repertoire for the subject matter they teach. As the teachers’ representational repertoire grows, it may improve teachers’ subject matter understanding (Wilson, Shulman & Richert, 1987). For, instance, when Even (1990) asked prospective secondary teachers to explain to the student in Algebra II how to graph the function \( f(x) = \frac{1}{x^2 - 1} \), half of the teachers said that first, they would emphasize a point wise approach. In their explanations, they would first suggest making a chart of some values of x and y, plot the points and then connect them in order to produce a smooth curve. The other half suggested that they would start looking first for undefined points to look at the behavior of the function. This example showed that prospective secondary teachers should know graphing of function is important after analyzing of some characteristics of function. They also should know that a good graph should not be based on the choice of numbers that are easy to compute, because some points can be more important than others. Even (1990) stated that prospective secondary teachers should be familiar with the main alternative approaches and their uses to make
appropriate choices between them. As the teachers actively create multiple representations, the students invent their own as they experience the representational activity of the teachers (Wilson, Shulman & Richert, 1987).

The teachers need to have the knowledge and understanding of the concept of function so that they should be able to help students to be flexible in their approach to functions. In addition, Even stated, “knowing what functions are and being able to work with them in different forms, representations and notations using appropriate ways, is important” (Even, 1990, p. 535).

**The strength of the concept**

*The strength of the concept* refers to the success of a concept in the discipline of mathematics. In this situation, the concept is very important, unique and open to new possibilities. Therefore, teachers should have a good understanding of the concept in the discipline of mathematics. They should fully understand/appreciate the importance of sub-topics or sub-concepts. Even (1990) stated that understanding of the concept of function must involve an understanding of the structure of functions and the inverse functions. The reason is that the inverse function cannot be understood in one simplistic way only. Understanding the inverse function as any other concept includes sub-topics or sub-concepts. In other words, understanding such sub-topics or sub-concepts requires teachers to have a good understanding of general meaning of concepts as well as the formal mathematical definition (Even, 1990).

**Basic repertoire**

*Basic repertoire* involves powerful examples that represent essential principles, properties, and theorems, etc. Gaining the basic repertoire gives insight into and a deeper understanding of general and more complicated knowledge for prospective secondary teachers. The reason is that teachers need to know and have easy access to specific examples. Even (1990) researcher stated, “Basic repertoire should be well known and familiar in order to be readily available for use” (p. 525). However, this does not mean that one should memorize and use it without understanding. The basic repertoire should be acquired meaningfully and used appropriately and wisely (Even, 1990).

The concept of function is a very important topic for prospective secondary teachers, since they need to teach the concept of function as a basic repertoire for their students. For instance, the prospective secondary teachers were asked why the graph of \( f(x) = ax^2 + bx + c \) looks down \( \cap \) when “a” (in the equation) is negative. Many of them could not explain why the
rule worked (Even, 1990). Memorizing the rule for quadratic functions gives no basis for
generalization. That is why teachers need to understand the relationship between the role of “a”
in the symbolic representation of a quadratic function and in the graphic representation.
Therefore, prospective secondary teachers should have a good understanding of the concept of
function and in particular, a basic repertoire.

**Different kinds of knowledge and understanding of function and mathematics**

Different kinds of knowledge and understanding of function and mathematics includes
both procedural and conceptual knowledge and the relationship between them. Procedural
knowledge involves formal language of mathematics and algorithms for completing
mathematical tasks. Conceptual knowledge should be learned meaningfully. This way, one
recognizes and generates relationships between units of knowledge. This aspect also involves
knowledge about mathematics. Knowledge about mathematics is a general knowledge about the
discipline that guides the construction and use of conceptual and procedural knowledge. This
way, prospective teacher recognizes and generates relationships between units of knowledge.
However, mathematical knowledge should include both procedural and conceptual knowledge
and the relationship between them (Even, 1990). For instance, when prospective secondary
teachers were asked to find the inverse function of \( f(x) = 10^x \), responses of prospective
secondary teachers showed that most of them did not use their procedural knowledge about
inverse of functions (Even, 1990). Instead they used their naive and immature conceptual
knowledge of what was an inverse function. Their answer for the root of the inverse function
was wrong. This example showed that one should generate a linkage between procedural and
conceptual knowledge to generate answers and understand what they are doing (Even, 1990).
Otherwise, when concepts and procedures are not connected, one may be able to solve problems,
or he/she may generate answers but not understand what they are doing. Especially in this
example, use of correct procedural knowledge could have helped prospective teachers to check
their naive conceptual knowledge.

The following aspect was utilized to analyze and describe the participants’ pedagogical
content knowledge.

**Analysis of students’ mistakes**

Analysis of students’ mistakes focuses on knowledge about common misconceptions.
The teacher should be able to determine whether a students’ answer is correct or incorrect. The
teacher should be able to help the students understand what is wrong and what is missing. In order to help the students, the teacher must understand sources of students’ mistakes. Knowledge about sources of the students’ helps the teacher understand the reasons for the students’ mistakes and make appropriate decisions (Even, 1989).

Similarly, by using her framework, Even (1993) examined of 162 prospective secondary teachers’ subject matter and pedagogical content knowledge of functions in her study. Even (1993) asked two kinds of questions: (1) the concept of function questions, and (2) hypothetical questions about teaching. The function questions focused on different representations and use of the concept. The hypothetical questions put a great emphasis on possible questions that a classroom teacher may face in actual teaching. Even (1993) found that participants did not have a modern conception of function. Their conceptions were similar to college students’ conceptions as described in the literature (Dreyfus & Eisenberg, 1983; Vinner & Dreyfus, 1989). Very few secondary prospective teachers could explain the importance and origin of the univalence requirement. Rather than explaining the meaning of a rule, most of the prospective teachers provided the rule for students to follow without understanding it (Even & Tirosh, 1995; Sanchez & Llinares, 2003).

In the same way, Wilson (1994) examined how a prospective secondary teacher understood the concept of function. In his study, Wilson used materials from a National Science Foundation- sponsored (NSF) project (Integrating Mathematics Pedagogy and Content in Teaching). The NSF materials focused on mathematical and pedagogical aspects of the function concept. The materials included questions to investigate prospective teacher’s understanding of the concept of function. The concept of function questions included graphical, algebraic representations of functions and non-functions, and real life applications of functions. These questions also consisted of an activity called, card sorting activity (Cooney, 1996). This activity focused on having the prospective teacher sorts several different functions (linear, quadratic, polynomial, exponential, logarithmic, and periodic, etc.) using four different representations (i.e., graphs, tables, verbal descriptions, and formulas). Although the prospective teacher’s understanding of function was dominated by a narrow view of mathematics at the beginning of the study, her understanding grew significantly. Her participation helped her see alternative ways of teaching mathematics. However, her perspective of mathematics and mathematics teaching was still relatively narrow at the end of the study.
Throughout the years, the criteria or starting point of the research studies has been differed for conceptions of teachers’ subject matter knowledge. In earlier studies, teacher’s subject matter knowledge was defined in quantitative terms by the number courses taken in college or teachers’ scores on standardized tests (e.g., Ball, 1991; Even, 1990, 1993, 1998). However, in recent years, other researchers analyzed and approached to teacher’s subject matter knowledge more qualitatively (e.g., Ball, 1991; Even, 1990, 1993, 1998; Wilson, 1992, 1994). These research studies focused on describing teacher’s subject matter knowledge and understanding of facts, concepts, and principles in the discipline, as well as analyzing what it means to know mathematics (Ball, 1991; Even, 1990, 1993, 1998; Shulman, 1986).

These research studies (Even, 1990, 1993, 1998; Vinner & Dreyfus, 1989; Wilson, 1994) on examining teacher’s subject matter knowledge and understanding of the concept of function used different materials to define what it means to know the concept of function in mathematics. For instance, in her study, Even (1990) asked function questions and hypothetical questions about teaching that focused on different representations and use of the concept of function. The hypothetical questions emphasized possible questions that a classroom teacher may face in classroom teaching. Vinner and Dreyfus (1989) used both strong graphical and strong non-graphical (representations or formulas) aspects of function concept. On the other hand, Wilson (1992, 1994) used materials from a National Science Foundation-sponsored (NSF) project (integrating Mathematics Pedagogy and Content in Teaching) that focused on mathematical and pedagogical aspects of the function concept. The concept of function questions included graphical, algebraic representations of functions and non-functions, and real life applications of functions. In these research studies, only Even (1990, 1993) examined teachers’ subject matter knowledge of the concept of function with both mathematical and hypothetical questions. By using seven aspects of her framework, Even (1990, 1993) analyzed teachers’ subject matter knowledge of the concept of function for teaching. However, these research studies did not focus on examining prospective secondary teachers’ subject matter knowledge of the concept of function in classroom instruction.

**Pedagogical Content Knowledge**

Shulman (2004) stated that pedagogical content knowledge “goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching” (p.203). In his
definition, Shulman (2004) observed that pedagogical understanding of subject matter should be included at the heart of definition of pedagogical knowledge.

Pedagogical content knowledge includes an understanding of what makes the learning of specific topic easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (Shulman, 2004, p. 203).

Shulman (2004) also stated that there are no single most powerful forms of representations. Therefore, the teacher must have at hand alternative forms of representation that derive from different sources such as research and textbook.

Within the category of pedagogical content knowledge, Shulman (2004) include, for the most regularly taught topics in one’s subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others (p.203).

Shulman defined teaching as a cycle that begins with an act of reason and continues with a process of reasoning. It ends with “performances of imparting, eliciting, involving, or enticing and is then thought about some more until the process can begin again” (p. 233). Therefore, Shulman (2004) described teaching as comprehension and reasoning, as transformation and reflection.

Fenstermacher (1978, 1986, cited in Shulman, 2004) stated that the focus of education is to educate teachers to reason about their teaching as well as to perform skillfully. In addition to reasoning, teachers must learn to use their knowledge base to provide the grounds for choices and actions (Shulman, 2004). Fenstermacher argued that good teaching must rely on a good foundation. However, the fact that the quality of teaching must rest on “the idea of influencing the grounds or reasons for teachers’ decisions places the emphasis precisely where it belongs: on the features of pedagogical reasoning that lead to or can be invoked to explain pedagogical actions” (Shulman, 2004, p.234). One’s understanding and knowledge of concept must be related to judgment and action and to appropriate uses of understanding in his/her pedagogical decisions (Shulman, 2004). However, pedagogical reasoning and action include a cycle (Shulman, 2004) through the activities of comprehension, transformation, instruction, evaluation, and reflection (Wilson, Shulman & Richert, 1987).
The following model of Wilson, Shulman & Richert (1987) examine teachers’ conception of pedagogical reasoning and action through six common aspects of the teaching: *comprehension, transformation, instruction, evaluation, reflection, and new comprehension* (See Figure 2.1).

![Figure 2.1: Model of Wilson, Shulman and Richert (1987)](image)

The Model of Wilson, Shulman & Richert (1987)

**Comprehension**

Pedagogical reasoning starts with *comprehension*. Teachers must critically understand a set of ideas, a piece of content, in terms of both its substantive and syntactic structure (Wilson, Shulman & Richert, 1987). The substantive structures involve the ideas, facts, and concepts of the field, as well as the relationships among those ideas, facts, and concepts. The syntactic structures include knowledge of the ways in which the discipline creates and evaluates new knowledge. Therefore, teachers “should understand how a given idea relates to other ideas within the same subject area and to ideas in other subjects as well” (Shulman, 2004, p. 235).
**Transformation**

The *transformation* includes sub-process: *interpretation, representation, adaptation, and tailoring* (Wilson, Shulman & Richert, 1987). However, later, Shulman (2004) stated that the transformation involves four stages: *preparation, representation, selection, adaptation and tailoring to student*. I believe the sub-process of the transformation is not very different from Shulman (2004)’s transformation stages. That is why I will incorporate previous four sub processes with four stages of Shulman (2004).

*Interpretation* involves reviewing instructional materials through one’s understanding of the subject matter. However, in his book called, *The Wisdom of Practice*, Shulman (2004) called this sub-process *preparation*. *Preparation* includes investigating and critically interpreting the materials of instruction. This preparation process involves detecting and correcting errors in the text, and the processes of structuring and segmenting the material into forms. Adaptation of the materials helps the teacher to have a better and more suitable understanding for teaching. As well as adaptation of the materials, the teacher grasps and extends his/her repertoire (instructional materials, programs, and conceptions).

When one examines the materials for instruction, he/she should possess a *representational repertoire* that consists of ideas (activities, illustrations, and examples) in the text or lesson (Shulman, 2004). Thinking through these key ideas helps the teacher to consider the alternative ways of representing subject matter to students (Shulman, 2004; Wilson, Shulman & Richert, 1987). In this way, the teacher uses his/her representational repertoire to transform the content for instruction.

*Adaptation* includes fitting the represented materials to the characteristics of the students in general. One needs to consider students’ misconceptions or misunderstandings about the material (student’s ability, student level). Choosing appropriate material is important because inappropriate material may interfere with students’ learning of the concept. For instance, when preparing a lesson plan on introducing the concept of function, one should consider students’ prior knowledge and how best to use that prior knowledge in developing a more sophisticated understanding of the material.

*Tailoring* refers to adapting materials to the specific students in one’s classroom instead of using the student population in general. In addition to adapting materials, the tailoring of instruction involves fitting representations to a group of a particular size (Shulman, 2004;
Wilson, Shulman & Richert, 1987). As a result of these processes of transformation, the teacher presents a lesson, unit, or course in a plan, or a set of strategies. Up to this point, this is a rehearsal for the performance of teaching. However, comprehension, transformation, evaluation, and reflection continue to occur during the teaching process.

**Instruction**

*Instruction* refers to an observable performance of the teacher in terms of effective teaching. It involves the most fundamental aspects of pedagogy such as organizing the classroom, assigning and checking work, interacting with students, and coordinating of learning activities, explanation, questioning, and discussion (Shulman, 2004; Wilson, Shulman & Richert, 1987). Shulman (2004) thought that the relationship between the comprehension of a new teacher and the styles of teaching employed is powerful. He said, “Teaching behavior is bound up with comprehension and transformation of understanding. Flexible and interactive teaching techniques are simply not available for the teacher, when she does not understand the topic to be used” (Shulman, 2004, p. 240).

**Evaluation**

*Evaluation* occurs both during and after instruction. Evaluation focuses on one’s own teaching and the lessons and materials employed in those activities (Shulman, 2004; Wilson, Shulman & Richert, 1987). The teacher checks his/her students’ understanding and misunderstanding, while teaching interactively. In order to determine students’ understanding of the lessons and materials, evaluation process requires a deep understanding of the material to be taught within the subject. This is a way to represent the use of pedagogical content knowledge (Shulman, 2004), which leads teachers directly to reflection.

**Reflection**

The teacher evaluates his/her instruction through *reflection* that he/she reviews and learns from his/her teaching experiences. Shulman (2004) said that the reflection process is a review of the teaching in comparison to the ends that teachers seek, and it could be done alone with the help of recording equipment or through memory. Reflection is a process where the teacher looks back at the teaching and learning that has occurred (Wilson, Shulman & Richert, 1987).

**New Comprehension**

*New comprehension* refers to having an increased awareness of the purpose of new instruction. “The teacher achieves new comprehension, both of the purposes and of the subjects
to be taught, and also of the students and of the processes and pedagogy themselves (Shulman, 2004, p. 241).” New comprehension improves the awareness of purposes of teacher’s instruction. It is also a new understanding that has been enhanced with the subject matter of instruction. New comprehension is an enriched understanding. It may grow slowly. On the other hand, a single experience may reach a quantum leap.

**Research Studies on Pedagogical Content Knowledge for Teaching Functions**

Pedagogical content knowledge of teachers has gained greater attention in recent years (e.g., Ball, 1988; Even, 1993; Shulman 1986; Wilson, Shulman & Richert, 1987). However, there has been little research conducted on prospective secondary teachers’ pedagogical content knowledge of the concept of function (Even, 1993, 1998; Even & Tirosh, 1995; Sanchez & Llinares, 2003). The studies (Even, 1993, 1998; Even & Tirosh, 1995; Sanchez & Llinares, 2003) that have examined the prospective teachers’ knowledge and their mathematical and pedagogical understanding help us to understand the role of a specific topic of subject matter in teacher learning (Cooney & Wilson, 1993).

In her study, Even (1993) examined 162 prospective secondary teachers’ subject matter knowledge and its relations with pedagogical content knowledge for teaching the concept of function. The prospective secondary teachers were asked to complete a questionnaire that included nine questions about functions and hypothetical questions (analyzing and responding to the students’ mistaken solutions or misunderstandings) related to teaching of the concept of function. Even (1993) designed this questionnaire based on the students’ limited conceptions and mistakes as described in the literature or known from her personal experiences with students. By using seven bodies of knowledge from her framework (Even, 1989), she analyzed prospective secondary teachers’ responses to the questionnaire. In this study, many of the prospective secondary teachers did not have a modern conception of functions. They held specific expectations about functions and their behavior. Many of the prospective teachers rejected unfamiliar functions. They used familiarity to accept non-function as functions. The prospective secondary teachers thought that familiar functions were or could always be represented as equations, formulas or graphs. The difference rested on those who understood the arbitrariness characteristic of functions and those who did not. Even (1993) found that many of the prospective secondary teachers did not have the appropriate concept image of a function. Similarly, in her following study, Even (1998) examined the prospective secondary teachers
interconnection between the flexibility in moving from one representation to another, and other aspects of knowledge and understanding with use of the questionnaire (Even, 1990). She found that the prospective secondary teachers had difficulties when they worked with different representations of functions.

In the same way, Even and Tirosh (1995) examined 162 prospective secondary teachers’ planned presentations of subject matter about functions in the last stage of their formal education at eight mid-western universities in the USA. In their study, Even and Tirosh (1995) focused on two main sources: (1) prospective teachers’ pedagogical content knowledge, and (2) prospective teachers’ sources of pedagogical content knowledge. Researchers investigated prospective secondary teachers’ pedagogical content knowledge, its sources, and their own experiences, both as learners and as teachers, and prospective secondary teachers’ sources of pedagogical content knowledge. For the sources of pedagogical content knowledge, Even and Tirosh (1995) looked at the depth of teachers’ own subject-matter knowledge of the material that they would use to teach. In addition, researchers dealt with two main sources of components of pedagogical content knowledge: knowledge about the subject matter and knowledge about the students. In order to do so, the prospective teachers were asked to respond to several interrelated aspects of function questions and hypothetical questions (i.e., analyzing or responding to students’ mistaken solutions or misunderstandings) from the study of Even (1990) to determine their instructional decisions. The study showed that most of the prospective teachers were familiar with the univalence requirement, and they included it in their definition of a function. Using the requirement as a criterion, these prospective teachers checked whether or not given mathematical objects were functions. However, their answers pointed out that prospective teachers did not really know or claim that the importance of univalence and the role of univalence were to distinguish between functions and non-functions. The prospective teachers did not recognize univalence as an arbitrary requirement. When the prospective teachers were asked to present the concept of function to the students, they relied on this easy rule (the vertical line test). Even and Tirosh (1995) stated that not knowing why the rule works affected the prospective teachers’ pedagogical content-specific choices. Many of the prospective teachers did not try to understand the sources of students’ responses. When they were asked directly, the prospective teachers had difficulty explaining why students reacted the way they did. The study concluded that prospective teachers’ subject matter knowledge influenced their organization of the content for
teaching. However, their subject matter knowledge was not much affected by their knowledge of students’ ways of thinking. Prospective teachers’ subject matter knowledge for two aspects (i.e., prospective teachers’ pedagogical content knowledge and prospective teachers’ sources of pedagogical content knowledge) was not sufficiently comprehensive and articulated for teaching (Even and Tirosh, 1995). Many of them did not make any attempt to understand the sources of students’ responses, because they had difficulty explaining why students reacted the way they did.

Sanchez and Llinares (2003) investigated the influence of four prospective secondary teachers’ ways of knowing the subject matter and images of mathematics teaching and learning on their hypothetical presentation of subject matter for teaching in the context of functions. This influence was examined in four interviews: (1) obtaining information related to their background and mathematics, (2) asking prospective secondary teachers to engage in practical tasks with associated textbook problems, (3) asking prospective secondary teachers to use the textbook problems in planning a hypothetical teaching sequence for the concept of function and provide arguments to justify their decisions, and (4) asking hypothetical situations from the study of Leinhardt, Zavlasky & Stein (1990) to prospective secondary teachers to diagnose students’ thinking and identify causes for the students’ response and to find a way of helping the students. In this context, Sanchez and Llinares (2003) used pedagogical reasoning (Shulman, 1986; Wilson, Shulman & Richert, 1987) as a theoretical construct to represent the transformation of content knowledge for the purpose of teaching. As a result of the study, three aspects became apparent: (1) their critical interpretation of the subject matter, (2) the representational repertoire they used, and (3) their adaptation of the subject matter. Sanchez and Llinares (2003) found that all four prospective teachers considered teaching as telling and learning as remembering. Researchers found a slight change on the influence of images of mathematics and teaching and learning on prospective secondary teachers’ organization of the subject matter for teaching. All four prospective teachers’ ways of knowing the concept of function influenced what they thought important for the students. The four prospective teachers looked at mathematics as a set of systematic procedures for solving problems. None of the prospective teachers provided information related to the pupils’ mathematical understanding for planning a hypothetical teaching sequence since they did not yet attend teaching practices.
The literature review showed that the research studies used different strategies or materials to examine prospective secondary teachers’ pedagogical content knowledge of the concept of function. Even (1993) used questions about functions and hypothetical questions for teaching. Similarly, Even and Tirosh (1995) selected questions related to functions (Even, 1990) and hypothetical questions. In the same way, Sanchez and Llinares (2003) used the concept of function questions from textbook and hypothetical questions. In addition to these questions, Sanchez and Llinares (2003) included a task that asked prospective secondary teachers to use the textbook problems in planning a hypothetical teaching sequence for the concept of function. However, the prospective secondary teachers had limited knowledge of students’ understanding, since they did not have teaching experience. Along the same lines as other studies (e.g., Even, 1993; Even & Tirosh, 1995), the study of Sanchez and Llinares (2003) pointed out the influence of subject matter knowledge for teaching on the process of pedagogical reasoning hypothetically, since the prospective secondary teachers did not attend teaching practice. However, only studies of Even (1989, 1990, 1993) and Sanchez and Llinares (2003) examined prospective secondary teachers’ pedagogical content knowledge on their hypothetical presentation of teaching. The following study expanded the set of studies in teaching and learning about functions. It went beyond administering questionnaires and conducting interviews by including an investigation of teaching. The study included a teaching episode in order to better understand the nature of relationship between the subject matter knowledge and pedagogical content knowledge of the concept of function.
CHAPTER 3

METHODOLOGY

The purpose of this study was to examine the dynamic interplay between pedagogical content knowledge and subject matter knowledge of prospective secondary mathematics teachers in the area of the concept of functions. Three research questions were examined. These were the following:

1. What is the nature of subject matter knowledge of prospective secondary mathematics teachers for teaching the concept of function?
   a) How do prospective secondary mathematics teachers classify relations into functions and non-functions?
   b) What are the prospective secondary mathematics teachers understanding of the relationships among different types of functions?
   c) How do the prospective secondary mathematics teachers translate from one representation to another?

2. What is the nature of pedagogical content knowledge of prospective secondary mathematics teachers for teaching the concept of function?

3. What is the nature of relationships between prospective secondary mathematics teachers’ subject matter knowledge and pedagogical content knowledge for teaching the concept of function?

The nature of the questions suggested that a qualitative research design would enable me to gather and analyze data through methods suitable to answer the questions.

Qualitative Study

Qualitative research is an approach where extensive description can be written in natural settings (Creswell, 2003). A qualitative research design is also flexible and less structured and it may be a developing design as the research progresses. This design allowed me to be open about what would be observed and collected in order to minimize missing something important (Wiersma, 2000).

Rossman and Rallis (1998) state that qualitative research is naturalistic and utilizes multiple methods. It is also emergent and interpretive. I selected the qualitative research design
because it was appropriate for the purpose of this study. I conducted the study in settings with which the participants were somewhat familiar. Through multiple data collection strategies participants were observed and data was gathered in order to answer the research questions. One framework was used to analyze prospective secondary teachers’ subject matter knowledge. A model was used to describe prospective secondary teachers’ pedagogical content knowledge of the concept of function.

A Multiple Case Study

Case studies are used to explore a program, an event, an activity, a process, or one or more individuals in depth (Creswell, 2003). For comparative reasons, the present study included multiple case studies of two prospective secondary mathematics teachers so that the results of these two cases could be compared and contrasted (Wiersma, 2000). Case study design was utilized to gain an in-depth understanding of the prospective secondary teachers’ knowledge of the concept of functions and its meaning for the researcher (Merriam, 1988). In this research study, the purposeful sampling was used to identify unique variations that have emerged in adapting to different conditions (Lincoln & Guba, 1985).

Participants

Purposeful sampling was used to select participants for this study. Wiersma (2000) states that the logic of purposeful sampling is based on a sample of information-rich cases that are studied in depth. There is no assumption that all members of the population are equivalent data sources, but those selected are believed to be information-rich cases.

This study was conducted with prospective secondary mathematics teachers who were enrolled in an undergraduate course “How Adolescences Learn Mathematics” in the Fall 2005 semester. In this class, there were thirty-three prospective teachers. The class was required for prospective teachers wanting to teach mathematics at the middle or high school level.

The prospective teachers were administered the function questionnaire (see Appendix A). Five criteria were used to select sixteen participants. These criteria were (1) input of the instructor of MAE 4330, (2) demographic information (i.e., teaching experience, academic status, grade point average, and mathematics course work completed) of the prospective teachers, (3) prospective teachers’ responses to the function questionnaire and card sorting activity, (4) willingness to participate and perceived ability to articulate thinking as determined by the instructors. The following briefly explains the method of selecting participants:
1. I interviewed the instructor of the MAE 4330 class, after each observation. The purpose of these interviews was to learn what she thought and observed about the role of the prospective teachers in the class activities. My goal was to learn what she thought about the prospective teachers that were willing to share and were open minded about sharing their ideas with their classmates. In this way, I could compare my observations with the observations of the instructor. With the help of this comparison of observations, I focused on the prospective teachers that participated more in class activities.

2. The demographics on the prospective teachers were used to look at their academic information such as grade point average, mathematics and methods courses. I used the demographic information to look at prospective teachers that had higher-level mathematics courses (i.e., linear algebra, calculus I and calculus II).

3. The prospective teachers’ responses to the function questionnaire (see Appendix A) and card sorting activity (see Appendix B) were used to choose the participants with strong mathematical understanding. The card sorting activity provided an opportunity for the prospective teachers to organize their knowledge about how common functions were typically represented (Wilson, 1992).

4. Based on the criteria given above, the prospective teachers who were willing to participate and had the ability to articulate their thinking were chosen for the purpose of this study.

**Methods of Data Collection**

I used various methods to collect data for this study, including semi-structured interviews, observations (i.e., field notes), documents (i.e., analysis of lesson plans and their teaching, and individually prepared lesson plan), and videotaped teaching episodes. The data were collected over six weeks. The function questionnaire (see Appendix A) was administered to the prospective teachers. After the analysis of their responses to the function questionnaire, sixteen prospective teachers were selected. Sixteen prospective teachers who answered most of the questions correctly were selected. The rest of the prospective teachers did not write anything or had incorrect answers. In the MAE 4330 class, these sixteen prospective teachers participated in a card sorting activity (see Appendix B), which was administered to the whole class. However, I looked closely at these sixteen prospective teachers. After an analysis of their responses in the card sorting activity Jack, Tim, Sara and Jane were selected for the purpose of this study. The purpose of this selection was to choose participants with strong mathematical
understanding of the concept of function. Jane decided not to participate due to her busy schedule. I eliminated Tim because his responses could not be studied in depth. In addition, his responses were not informative rich case. After selecting the participants, two sixty-minute interviews were conducted with two prospective secondary teachers about the function questionnaire. Another sixty-minute interview was conducted with the participants about the card sorting activity. The participants were given two lesson plans (see Appendix C) to analyze as a part of a class activity. However, analysis of each lesson plan took place in one-class period. After the analysis of the lesson plans, the participants watched videos of lessons taught by Mr. A (the liberal arts mathematics teacher) to students at the Southern High School. These lessons focused on probability. In the class, along with their classmates, these two participants discussed and shared their ideas about the lessons and the teaching of the lessons. I observed the discussion and took field notes. A sixty-minute interview was conducted to discuss these lesson plans and teaching of them.

After the discussion, the participants were given a lesson plan guideline (see Appendix D) that included student objectives. The lesson plan focused on how to introduce the exponential function and teach it to a small group of senior high school students at the Southern High School. The participants’ lesson plan was collected and analyzed. After the analysis of the lesson plan, a sixty-minute interview was conducted with the participants. Its purpose was to ask the participants’ questions about their lesson plan and the analysis of the two lesson plans.

After the interview, the participants taught their lesson to a small group of senior high school at the Southern High School. Their teaching was videotaped. After his or her teaching, a sixty-minute interview was conducted with each participant individually. During this interview, each participant watched a video clip of his or her teaching, and was asked questions about the teaching.

**Interviews**

Interviews are a major source of qualititative data that are needed for understanding the phenomenon under study (Merriam, 1988). Creswell (2003) states that interviews allow the researcher to have some control of the line of questioning. For Lincoln and Guba (1985, p. 273), “A major advantage of the interviews is that it permits the respondent to move back and forth in time-to reconstruct the past, interpret the present, and predict the future, all without leaving a comfortable chair.”
For this study, a semistructured interview technique was employed. Six interviews were conducted with each participant. Each interview was one hour long and included questions that focused on the participants’ responses to the function questionnaire (see Appendix A) and the card sorting activity (see Appendix B), analysis of two lesson plans (see Appendix C and D) and teaching of these lessons, a lesson plan on exponential functions (see Appendix E) and participants’ reflections about teaching.

Merriam (1988) suggested that the key to having good data from interviewing is to ask good questions. I have prepared the function questionnaire and card sorting activity (see Appendix A and B) to engage students in thinking and encourage them to describe tasks. I also asked them to elaborate on their solutions during the interviews. The tasks and questions were planned in advance, but I was flexible and followed up on the participants’ responses. These interviews included questions that were designed to elicit views and opinions of the participants (Creswell, 2003). These interviews were arranged at the participants’ convenience and availability during the study.

Although interviews were useful for me to observe the participants directly, there were difficulties involved in interviewing that need to be addressed; such as participants’ “uneasiness with being recorded is the drawback” (Merriam, 1988, p. 81). I observed the MAE 4330 class and interviewed them throughout the semester so that the participants were comfortable and got used to the fact that they were being studied or audio- and video-taped during interviews. Furthermore, I kept in mind that “all information obtained from participants has been selected, either consciously or unconsciously, from all that he or she knows. What you get in an interview is simply the participant’s perception” (Merriam, 1988, p. 84).

Observations

Lincoln and Guba (1985) suggest that field notes taken during the interviews and field notes from observations can be transcribed and analyzed for preliminary units and categories of information that can be checked, expanded, and related during subsequent observations.

As an outsider an observer will notice things that have become routine to the participants themselves, things which may lead to understanding the context. The participant observer gets to see the things firsthand and to use his/her knowledge and expertise in interpreting what is observed, rather than relying upon once- removed accounts from interviews (Merriam, 1988, p. 88).
During classroom observations I acted as a researcher participant (Gans, 1982, cited in Merriam, 1988). Through this role I was partially involved in the class while still functioning as a researcher. Merriam (1988) cautions that the researcher participant affects the environment that is being observed. Ideally, the researcher’s role as a neutral participant should not change the situation in any way that might lead to compromising the data. In reality, a primary activity of the researcher participant is to monitor and document these interactions (e.g., through field notes) in order that during the data interpretation phase the researcher is able to take these into consideration.

Once one has become familiar with the setting and begins to see what is there to observe, serious data collection can begin. “It takes a great concentration to observe intently, remember as much as possible, and then record, in as much detail as possible, what one has observed” (Merriam, 1988, p. 91).

During my classroom observations, I took the preceding difficulties into account in order to avoid encountering possible problems. I took descriptive field notes about participants’ behaviors and activities twice a week in the classroom. In addition, after my observation, I interviewed the instructor of the MAE 4330 class for 10-15 minutes to have her opinion about the participants’ activities in the classroom.

I also tried to verify what emerged in the interviews. When I reached the point where I saw a repetition of the participants’ responses in the class activities and noticed that they used their knowledge and expertise in interpreting what was observed, I stopped observing the class in order to ensure reliability of data collected and proceed with the subsequent analyses.

**Documents**

Documents consisted of an individually prepared lesson plan and math autobiography. There were two major limitations stemming from documentary data: (a) the data may be incomplete from the researcher’s point of view when it is not developed for research purposes, and (b) misrepresentation of personal documents might be unintentional, because the participant is unaware of his or her biases or simply does not remember accurately (Merriam, 1988). I took the limitations of the documentary data into consideration when the documents were analyzed and tried to determine accuracy of the document as well as its reasons and the circumstance in which it was written.

**Videotaping Interviews**
Interviews were audio-and videotaped in order to observe participants’ behaviors during interviews and teaching episodes. Each interview took one hour and included questions that focused on the participants’ responses to the function questionnaire, the card sorting activity, analysis of two lesson plans, their lesson plan and reflection about their teaching. Their participation involved six interviews. For Roschelle (2000, p. 727), video offers the following advantages:

- Video offers repeated viewing of the same event.
- Video can support interpretations from many points of view.
- Video can be shared with the participants, thus creating the opportunity for getting a participants’ own point of view about his or her behaviors.

**Procedure**

I collected the data through four stages: (1) the function questionnaire, (2) card sorting activity, (3) analysis of two lesson plans and their teaching, and (4) preparation and analysis of lesson plans on exponential functions and video teaching episode. The participants, Jack and Sara, were interviewed individually after every stage in this study.

Jack and Sara took the function questionnaire in class. They were given 60 minutes to complete it. After analyzing their responses to the function questionnaire, I conducted two sixty-minute interviews with each participant. During the interviews, I asked them to elaborate on their answers to the function questionnaire. The second stage of the study employed a card sorting activity (Cooney, 1992). The participants received twenty-eight different examples of seven types of functions (e.g., linear, quadratic, polynomial) given in four different representations (i.e., verbal, table, graph, equation). Each function was written on an index card and the participants were asked to determine different ways of sorting the functions using certain criteria (e.g., function type, different representations). The participants were given sixty minutes to complete it. Having analyzed their responses to the card sorting activity, I conducted a sixty-minute interview with each participant regarding their responses.

In the third stage of the study, since they did not have any experience in designing a lesson plan, I wanted the participants to analyze two lesson plans and teaching videos of these lessons. One lesson plan and its teaching was written and implemented as a bad example and the other lesson plan and its teaching was written and implemented as a good example. The participants were asked to analyze them and determine which lesson plan and teaching was better.
and why it was better. A sixty-minute interview was conducted to discuss these lesson plans and the teaching of them.

In the fourth stage of the study, the participants were given a lesson plan guideline and objectives and asked to write a lesson plan on exponential functions. A sixty-minute interview was conducted to discuss the participants’ lesson plans. When each participant was teaching his/her lesson to a group of high school seniors at the Southern High School, their classes were videotaped by the researcher. The two teaching episodes were analyzed and a sixty-minute interview was conducted individually with each participant to give them an opportunity to evaluate and reflect on his/her teaching of the lesson.

In this study, Even’s (1989) framework was used to examine the participants’ subject matter knowledge and pedagogical content knowledge of the concept of function and analyze their responses to the function questionnaire and card sorting activity. Even (1989) examined prospective teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function through seven aspects of subject matter knowledge for teaching functions: essential features, different representations, the strength of the concept, basic repertoire, alternative ways of approaching, different kinds of knowledge and understanding of function and mathematics, and analysis of students’ mistakes.

**Essential features** deal with two important concepts of a function, univalence and arbitrariness. The univalence requirement is that for each element in the domain there is only one element in the range. Arbitrariness of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. Arbitrariness does not have to be defined by any specific expression, follow some regularity, or be described by a graph with any particular shape.

**Different representations** means that one understands the concept in different illustrations and is able to interpret and form connections among and between them.

**The strength of the concept** refers to teachers’ understanding of characteristics of the concept. Even (1990) states that understanding of the concept of function must involve an understanding of the structure of functions and inverse functions.

**Basic repertoire** involves the most powerful examples (e.g., essential principles, properties) that give insight into and a deeper understanding of general and more complicated knowledge for the participants.
Alternative ways of approaching refers to different uses of the concept in the different divisions of mathematics, other disciplines, or everyday life. The alternative ways are different from the other, and none are suitable for all situations. The teachers need to have the knowledge and understanding of the concept of function so that they can help students to be flexible in their approach to functions.

Different kinds of knowledge and understanding of functions and mathematics includes both procedural and conceptual knowledge and the relationship between them. Procedural knowledge involves the formal language of mathematics and algorithms for completing mathematical tasks. Conceptual knowledge should be learned meaningfully. This way, one recognizes and generates relationships between units of knowledge. This aspect also involves knowledge about mathematics. Knowledge about mathematics is a general knowledge about the discipline that guides the construction and use of conceptual and procedural knowledge.

The following aspect was utilized to analyze and describe the participants’ pedagogical content knowledge.

Analysis of students’ mistakes focuses on knowledge about common misconceptions. The teacher should be able to determine whether a student’s answer is correct or incorrect. The teacher should be able to help the student understand what is wrong and what is missing. In order to help students, the teacher must understand sources of students’ mistakes. Such knowledge helps the teacher understand the reasons for the students’ mistakes in order to make appropriate decisions (Even, 1989).

The questions in the function questionnaire include different types of functions (e.g., linear, quadratic, exponential, logarithmic), definition of a function and non-function, and different representations of functions (e.g., graphical, equation, set notation). These questions were utilized to examine prospective teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function. To examine the prospective teachers’ pedagogical content knowledge of function, Even (1989) asked the prospective teachers to give an alternative definition of a function and to analyze students’ incorrect solutions. I believe that having the participants answer these questions and analyze students’ incorrect solutions is valuable and informative but not sufficient to examine the participants’ pedagogical content knowledge. Their analysis of students’ mistakes in hypothetical situations would give me an idea of how they would approach the problem and what they would do to overcome students’ difficulties.
However, it is still difficult to predict how the prospective teachers would deal with students’ incorrect solutions in a real classroom environment.

In addition, Even’s (1989) framework gives a general overview of prospective teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function, which is very necessary to the study. However, I wanted to analyze the relationship between subject matter knowledge and pedagogical content knowledge of functions in the case of exponential functions. As a result, I felt the necessity to utilize the model suggested by Wilson, Shulman and Richert (1987) to examine the relationship between subject matter knowledge and pedagogical content knowledge as well as the growth in understanding of the content for teaching as a result of these tasks. In other words, I used Even’s framework to examine the participants’ subject matter knowledge of the concept of function through the mathematics questionnaire and card sorting activity as well as their pedagogical content knowledge through analysis of students’ incorrect solutions.

Wilson, Shulman and Richert’s (1987) model enabled me to investigate participants’ growth in content and pedagogical content knowledge as well as improvement in their understanding of the content for teaching as a result of these tasks (i.e., analyzing two lesson plans, preparing a lesson and video teaching of the lesson on exponential functions).

I conducted this study in four stages (the function questionnaire, card sorting activity, analysis of two lesson plans and their teaching, and preparation and analysis of lesson plans on exponential functions and video teaching episode). After describing the participants’ subject matter knowledge and pedagogical content knowledge based on their responses to the questions in the function questionnaire and card sorting activity, I tried to describe the participants’ subject matter knowledge and pedagogical content knowledge of exponential functions under six aspects suggested by Wilson, Shulman and Richert (1987); comprehension, transformation, instruction, evaluation, reflection, and new comprehension. The model of Wilson, Shulman and Richert (1987) helped me to organize the tasks (i.e., analysis of two lesson plans and their teaching, and preparation and analysis of lesson plans on exponential functions and video teaching episode).

At the end of this model of Wilson, Shulman and Richert (1987), I examined whether any change occurred on the participants’ perspective for teaching as a result of doing these tasks. In other words, I wanted to investigate whether or not participants improved their pedagogical content knowledge as a result of doing these tasks.
Using this model, I investigated the nature of pedagogical content knowledge of participants for teaching the concept of function. I looked at how the participants used different forms of representations, ideas, examples, activities and explanations in their lesson plans. When they taught their lesson on exponential functions and how they made it comprehensible to the group of high school seniors, I examined whether or not the participants recognized what made the learning of exponential functions easy or difficult for the group of high school seniors. I investigated whether or not the participants took into consideration the group of high school seniors’ current understanding to make instructional decisions. I looked at how the participants differentiated what the students comprehended and what they could perform (Graber, 1999), as well as how they provided the grounds for choices and actions. I also investigated the nature of relationships between the participants’ subject matter knowledge and pedagogical content knowledge for teaching the concept of function. I looked at how the participants used their subject matter knowledge of the concept of function for teaching.

**Comprehension** includes both substantive and syntactic structure. The substantive structures involve the ideas, facts, and concepts of the field, as well as the relationships among those ideas, facts, and concepts. The syntactic structures include knowledge of the ways in which the discipline creates and evaluates new knowledge. One should know what exponential function is and how it is defined as well as how exponential functions are related to logarithmic functions. In addition, one should know different representations (equation, graph, table and verbal) to describe exponential functions and to make transitions from one representation to another.

**Transformation** includes sub-processes: *interpretation, representation, adaptation, and tailoring*. *Interpretation* involves reviewing instructional materials from one’s understanding of the subject matter. *Preparation* includes investigating and critically interpreting the materials of instruction. This preparation process involves detecting and correcting errors in the text, and the processes of structuring and segmenting the material into forms. Adaptation of the materials helps teachers have a better and more suitable understanding of teaching. During this process, a teacher grasps and extends his/her repertoire (i.e., instructional strategies, programs, and conceptions). When a teacher examines the materials for instruction, he/she should possess a *representational repertoire* that consists of ideas (e.g., activities, illustrations, and examples) in a lesson (Shulman, 2004). Thinking through these key ideas helps a teacher consider the
alternative ways of representing subject matter to students (Shulman, 2004; Wilson, Shulman & Richert, 1987). Adaptation includes fitting the represented materials to the characteristics of the students in general. One needs to consider students’ misconceptions or misunderstandings about the material (student’s ability, student level). Choosing appropriate materials is important because inappropriate materials may interfere with students’ learning of the concept. Tailoring refers to adapting materials to the specific student in one’s classroom instead of using materials for the student population in general.

In this study, the participants analyzed two lessons about probability. They looked at how these lesson plans were prepared and taught. The participants were asked to compare the lesson plans and determine which one was better. During the analysis of the two lessons, the participants went through the transformation process, even though they did not prepare or teach these lessons. The participants had to look at the sub-processes of transformation to analyze these lessons. In addition, the participants prepared a lesson plan on exponential function for a small group of students. When the participants were asked to prepare a lesson, they used their subject matter knowledge and pedagogical content knowledge of the exponential functions. Then, the participants looked for different examples and activities for this lesson. They tried to choose the most appropriate examples and activities in their lesson plans and tried to adapt examples and activities that would be appropriate for the students’ needs.

Instruction refers to observable performance of the teacher in terms of effective teaching. It involves the most fundamental aspects of pedagogy, such as organizing the classroom, assigning and checking work, interacting with students, coordination of learning activities, explanation, questioning, and discussion (Shulman, 2004; Wilson, Shulman & Richert, 1987). When the participants analyzed two lessons, they analyzed how these lessons were taught by an experienced mathematics teacher (Mr. A) as well as the performance of the teacher. In addition, the participants prepared a lesson plan on exponential functions and taught their lessons to a group of high school seniors. I videotaped their teaching to analyze their performance. I examined how the participants utilized their subject matter knowledge and pedagogical content knowledge of exponential functions to teach the lesson.

Evaluation occurs during instruction. The teacher checks both understanding and misunderstanding of students as he/she is teaching. Checking the understanding of students requires all forms of the participants’ comprehension and transformation. Evaluation is also
directed at the teacher’s teaching methods and at the lesson. I utilized the videos of the participants’ teaching for evaluation. After their classroom teaching of the lessons on exponential functions, I conducted an interview with each participant. They watched the video clips and evaluated their teaching. They looked at what the group of high school seniors understood or misunderstood as a result of the teaching of exponential functions.

The teacher evaluates his/her instruction through reflection. Reflection is a process where by the teacher looks back at his/her teaching that has occurred. He/ she recaptures the events and the accomplishments. In this study, reflection was done with the help of video teaching of the lesson plan on exponential functions. The participants watched video clips and then looked back at their teaching. They reviewed their forty-minute teaching from the beginning to the end of the lesson.

New comprehension refers to having an increased awareness of the purpose of new instruction. Having collected the data and conducted the interviews, I looked at whether or not the participants changed their perspective for teaching as a result of completing these tasks. I looked at whether or not the nature of their pedagogical content knowledge for teaching the concept of function changed as a result of completing these tasks.

Data Sets

In the following section, I will provide descriptions of the lesson plans, teaching of these lessons, the participants’ lesson plans on exponential functions, and their teaching of exponential functions as well as my observations.

I used these descriptions to analyze and compare the participants’ responses during the interviews in this study. In addition, I wanted to provide some information for readers so that they would see what was done and how the data was analyzed in this study.

Function Questionnaire

The function questionnaire adapted from the studies of Even (1989) and Wilson (1992) included nineteen mathematics problems addressing different aspects of subject matter knowledge (e.g., examples of functions and non-functions, different ways of approaching functions) and analysis of students’ incorrect solutions (see Appendix A). I did not intend to measure the performance and success of the participants with this questionnaire. Its purpose was to describe the nature of the participants’ pedagogical and content knowledge. The function questionnaire included different types of functions (e.g., linear, quadratic, exponential,
logarithmic), definition of function and non-function, and different representations of functions (e.g., graphical, equation, set notation).

I asked the participants to show their method of solution and explain their answers so that I could examine the nature of their subject matter knowledge. In addition, I also asked them to explain their solution strategies for students’ errors so that I could investigate the nature of their pedagogical content knowledge of the concept of function. In their responses, I looked at how the participants classify relations into functions and non-functions. I examined their understanding of different types of functions. I looked at how the participants used different representations and translated from one representation to another.

**Card Sorting**

The card sorting activity (Cooney, 1996) included twenty-eight different examples of seven types of functions (i.e., linear, quadratic, polynomial, exponential, logarithmic and trigonometric and rational functions) given in four different representations (i.e., tables, graphs, equations, and verbal descriptions) (see Appendix B). Each card was written on an index card and participants were asked to arrange index cards of functions into piles using both different representations and classification of functions. In the card sorting activity, the participants were asked seven questions and organized their knowledge about seven different categories of functions using different representations.

The card sorting activity helped me determine how the participants utilized different representations and how they moved from one representation to another. In addition, the card sorting activity was employed for assessing participants’ understanding of the relationships among different types of functions. I looked at how they identified the seven types of functions using four different representations.

**Analysis of Two Lesson Plans and Their Teaching Episodes**

At the time of the study, the participants, Jack and Sara, were enrolled in MAE 4330, a course in which prospective teachers had analyzed middle and high school mathematics curricula from the perspective of adolescents’ learning of mathematics. They also discussed theoretical and practical aspects of learning and teaching mathematics in this class. Jack and Sara did not have any experience in writing a lesson plan so my major professor, the instructor of the MAE 4330 course, and I wanted the participants to gain some experience in preparing lesson plans as
well as analyzing video teaching episodes of those lessons before they wrote a lesson plan and taught it for this study.

We asked Mr. A at Southern High School to participate in this study. Mr. A, had a bachelor degree and master degree in mathematics education and was pursuing his Ph.D. in mathematics education. Mr. A taught two years at the middle school level and one year at the high school level. He had been teaching undergraduate methods courses for one year.

Mr. A agreed that he would prepare two lesson plans. The first lesson was on *Theoretical and Experimental Probability* (see Appendix C). Instead of preparing the lesson plan before implementing it, Mr. A taught the lesson unprepared, and then prepared the lesson plan. In other words, the lesson was not prepared prior to teaching. The second plan was on *Fundamental Counting Principles* (see Appendix D) and was written before he taught the lesson.

I videotaped Mr. A’s teaching of the lessons. I first gave the participants the lesson plans, then the videos of the teaching of those lesson plans. Later, I met with Mr. A to talk about the classes he taught for the study. I video-recorded the interview with Mr. A so that I could have the participants watch this video as well as describe his teaching from my point of view when I analyzed the data of the study. The prospective teachers in MAE 4330 watched the videos of the teaching of those lesson plans and discussed what they thought about Mr. A implementation of these two lessons. One sixty minute interview was conducted with each of the participants to discuss what they thought about the lesson plans as well as which lesson plan they found better and why.

**A Lesson Plan on Exponential Functions**

In this study, the participants did not have any experience in writing a lesson plan so my major professor, the instructor of the MAE 4330 course, and I decided to give a lesson plan guideline including objectives to the participants.

This way, the participants prepared the lesson. Their preparation focused on goals and procedures of the lesson. The decision to give the objectives of the lesson was supported by my committee members as well. Without objectives, the committee suggested that the participants prepare a lesson that might be related to exponential function, but it would not be designed for teaching of exponential functions. In addition, the committee suggested that the participants should be kept in the same direction so that I could compare their lessons.

**Teaching Episode**
With the analysis of lesson plans and teaching videos of these lessons, the participants gained some experience in determining which of these lessons were prepared and taught better. Then, the participants were given a lesson plan guideline with objectives and asked to write a lesson plan on exponential functions. The purpose of having participants write a lesson was to investigate how they used their subject matter knowledge and pedagogical content knowledge for preparing a lesson plan and teaching. I examined the nature of the relationships between the participants’ subject matter knowledge and pedagogical content knowledge for teaching exponential functions.

I video-recorded their teaching so that I could investigate the nature of pedagogical content knowledge of participants for teaching the exponential functions. I had participants watch video clips of their teaching evaluation and reflect on their implementation of the lessons.

During the interview with each participant, I asked them what they thought about their teaching and whether any change occurred in their thinking about how they should teach the exponential function. This enabled me to examine how the participants transformed their subject matter knowledge for teaching.

Analysis was ongoing starting from the beginning of data collection. Analyses of each individual participant occurred along with analyses across the cases. This enabled me to obtain insight on the participants’ subject matter knowledge and pedagogical content knowledge of the concept of function.

My goal was to provide rich descriptions to determine the nature of the participants’ subject matter knowledge and pedagogical content knowledge of the concept of the functions. Rich descriptions of the data enabled me in making a transfer to reach a conclusion (Lincoln & Guba, 1985). The data (participants’ responses to the function questionnaire and the card sorting activity, semistructured interviews, observations (i.e., field notes), documents (i.e., individually prepared lesson plan), and videotaped teaching episodes were analyzed, as they were collected.

**Triangulation**
As the study progresses, each data piece was used to triangulate each data piece against at least two other sources (Lincoln & Guba, 1985). I also compared and contrasted the participants who were different or similar in their subject matter knowledge and pedagogical content knowledge as they prepared to teach their lesson. For example, when assertion was aroused from participants’ response to questions about the concept of function, it was compared to data from other resources such as field notes, transcribed interviews and video taped teaching episodes. I did not give serious consideration to any information that could not be triangulated.

**Peer Debriefing**

In this proposed study, peer debriefing was used for multiple purposes (Lincoln & Guba, 1985): (1) establishing credibility, (2) testing hypotheses in the inquirer’s mind, (3) developing the next step in the emerging methodological design, and (4) opportunity for clearing my mind.

A debriefing process is helpful for establishing credibility. A debriefing process with my major professor helped me search out questions such as researcher’s biases, basis for interpretations of data sources and its meanings. The debriefing process took place with my major professor opinion on comments made by the participants that reflected their subject matter knowledge and pedagogical content knowledge of the concept of function. Since my major professor was also the instructor of MAE 4330 class that I collected my data, the debriefing process provided me with an opportunity to test working hypotheses that may be emerging in my mind and helped me develop and test the next steps in the emerging methodological design. The process helped me to clear my mind from emotions and feelings that may be preventing the next step in this study.

**Member Checking**
Member check is the most essential technique for establishing the credibility of the study (Lincoln & Guba, 1985). In this study, data, interpretations, and conclusions were tested with the participants. The participants were given the opportunity to react to the interpretations of the data throughout the investigation. This enabled me to look at what they knew about functions. According to Lincoln and Guba (1985), immediate member checking has several benefits:

- It provides the opportunity for the participants to respond to what he/she intend by acting in a certain way.
- It provides the participants the opportunity to correct errors and challenge how errors recognized.
- It gives the participants the opportunity to volunteer additional information.
- It provides the researcher the opportunity to summarize the data analysis.
- It gives the participants the opportunity to confirm the overall adequacy of data.

The following paragraph includes a summary of chapter 3-methodology, and what chapter 4-results focusing on explaining case studies of three participants Jack and Sara.

In the following chapter 4, I focus on case studies of two participants, Jack and Sara. I examined each case individually and looked closely at the nature of each participant’s subject matter knowledge and pedagogical content knowledge for teaching the concept of function. In addition, I looked at the nature of the relationships between participant’s subject matter knowledge and pedagogical content knowledge for teaching the concept of function.
CHAPTER 4

FINDINGS

This study involved multiple case studies of two college students who were enrolled in a MAE 4330 course during the Fall 2005 semester. The findings of the case studies are reported in this chapter. My goal was to answer the following research questions:

1. What is the nature of subject matter knowledge of prospective secondary mathematics teachers for teaching the concept of function?
   a) How do prospective secondary mathematics teachers classify relations into functions and non-functions?
   b) What are the prospective secondary mathematics teachers’ understanding of the relationships among different types of functions?
   c) How do the prospective secondary mathematics teachers transfer from one representation to another?

2. What is the nature of pedagogical content knowledge of prospective secondary mathematics teachers for teaching the concept of function?

3. What is the nature of the relationships between prospective secondary mathematics teachers’ subject matter knowledge and pedagogical content knowledge for teaching the concept of function?

Data Collection Procedure

I collected the data through four stages: (1) the function questionnaire, (2) card sorting activity, (3) analysis of two lesson plans and their teaching, and (4) preparation and analysis of lesson plans on exponential function and video teaching episode. The participants, Jack and Sara, were interviewed individually after every stage in this study.

Jack and Sara took the function questionnaire in class. They were given 60 minutes to complete it. After analyzing their responses to the function questionnaire, I conducted two sixty-minute interviews with each participant. During the interviews, I asked them to elaborate on their answers to the function questionnaire. The second stage of the study employed a card sorting activity (Cooney, 1992). The participants received twenty-eight different examples of seven types of functions (e.g., linear, quadratic, polynomial) given in four different
representations (i.e., verbal, table, graph, equation). Each function was written on an index card and the participants were asked to determine different ways of sorting the functions using certain criteria (e.g., function type, different representations). The participants were given sixty-minutes to complete the activity in the class. Having analyzed their responses in card sorting activity, I conducted a sixty-minute interview with each participant regarding their responses.

In the third stage of the study, since they did not have any experience in planning a lesson plan, I wanted the participants to analyze two lesson plans and teaching videos of these lessons. One lesson plan and its teaching was written and implemented as a bad example and the other lesson plan and its teaching was written and implemented as a good example. The participants were asked to analyze them and determine which lesson plan and teaching was better and why it was better. A sixty-minute interview was conducted to discuss these lesson plans and teaching of them.

In the fourth stage of the study, the participants were given a lesson plan guideline and objectives and asked to write a lesson plan on exponential functions. A sixty-minute interview was conducted to discuss the participants’ lesson plans. When each participant was teaching his/her lesson to a group of high school seniors at the Southern High School, their classes were videotaped by the researcher. The two teaching episodes were analyzed and a sixty-minute interview was conducted individually with each participant to give them an opportunity to evaluate and reflect on his/her teaching of the lesson. The following figure illustrates the order of the data collection procedure (see Figure 4.1).
In this study, Even's (1989) framework was used to examine the participants’ subject matter knowledge and pedagogical content knowledge of the concept of function and analyze their responses to the function questionnaire and card sorting activity. Even (1989) examined prospective teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function through seven aspects of subject matter knowledge for teaching functions: essential features, different representations, the strength of the concept, basic repertoire, alternative ways of approaching, different kinds of knowledge and understanding of function and mathematics, and analysis of students’ mistakes.

Essential features deal with two important concepts of a function, univalence and arbitrariness. The univalence requirement is that for each element in the domain there is only one element in the range. Arbitrariness of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. Arbitrariness does not have to be defined by any specific expression, follow some regularity, or be described by a graph with
any particular shape.

*Different representations* mean that one understands the concept in different illustrations and is able to interpret and form connections among and between them.

*The strength of the concept* refers to teachers’ understanding of characteristics of the concept. Even (1990) states that understanding of the concept of function must involve an understanding of the structure of functions and inverse functions.

*Basic repertoire* involves the most powerful examples (e.g., essential principles, properties) that give insight into and a deeper understanding of general and more complicated knowledge for the participants.

*Alternative ways of approaching* refers to different uses of the concept in the different divisions of mathematics, other disciplines, or everyday life. The alternative way is different from the other, and none are suitable for all situations. The teachers need to have the knowledge and understanding of the concept of function so that they can help students to be flexible in their approach to functions.

*Different kinds of knowledge and understanding of function and mathematics* includes both procedural and conceptual knowledge and the relationship between them. Procedural knowledge involves the formal language of mathematics and algorithms for completing mathematical tasks. Conceptual knowledge should be learned meaningfully. This way, one recognizes and generates relationships between units of knowledge. This aspect also involves knowledge about mathematics. Knowledge about mathematics is a general knowledge about the discipline that guides the construction and use of conceptual and procedural knowledge.

The following aspect was utilized to analyze and describe the participants’ pedagogical content knowledge.

*Analysis of students’ mistakes* focuses on knowledge about common misconceptions. The teacher should be able to determine whether a student’s answer is correct or incorrect. The teacher should be able to help the student understand what is wrong and what is missing. In order to help students, the teacher must understand sources of student’s mistakes. Such knowledge helps the teacher understand the reasons for the students’ mistakes in order to make appropriate decisions (Even, 1989).

The questions in the function questionnaire include different types of functions (e.g., linear, quadratic, exponential, logarithmic), definition of a function and non-function, and
different representations of functions (e.g., graphical, equation, set notation). These questions were utilized to examine prospective teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function. To examine the prospective teachers’ pedagogical content knowledge of function, Even (1989) asked the prospective teachers to give an alternative definition of a function and to analyze students’ incorrect solutions. I believe that having the participants answer these questions and analyze students’ incorrect solutions is valuable and informative but not sufficient to examine the participants’ pedagogical content knowledge. Their analysis of students’ mistakes in hypothetical situations would give me an idea of how they would approach the problem and what they would do to overcome students’ difficulties. However, it is still difficult to predict how the prospective teachers would deal with students’ incorrect solutions in a real classroom environment.

In addition, Even’s (1989) framework gives a general overview of prospective teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function, which is very necessary to the study. However, I wanted to analyze the relationship between subject matter knowledge and pedagogical content knowledge of functions in the case of exponential functions. As a result, I felt the necessity to utilize the model suggested by Wilson, Shulman and Richert (1987) to examine the relationship between subject matter knowledge and pedagogical content knowledge as well as the growth in understanding of the content for teaching as a result of the tasks. In other words, I used Even’s framework to examine the participants’ subject matter knowledge of the concept of function through the function questionnaire and the card sorting activity as well as their pedagogical content knowledge through analysis of students’ incorrect solutions.

Wilson, Shulman and Richert’s (1987) model enabled me to investigate participants’ growth in content and pedagogical content knowledge as well as improvement in their understanding of the content for teaching as a result of these tasks (i.e., analyzing two lesson plans, preparing a lesson and video teaching of the lesson on exponential functions).

I conducted this study in four stages (the function questionnaire, card sorting activity, analysis of two lesson plans and their teaching, and preparation and analysis of lesson plans on exponential function and video teaching episode). After describing the participants’ subject matter knowledge and pedagogical content knowledge based on their responses to the questions in the function questionnaire and card sorting activity, I tried to describe the participants’ subject
matter knowledge and pedagogical content knowledge of exponential functions under six aspects suggested by Wilson, Shulman and Richert (1987): comprehension, transformation, instruction, evaluation, reflection, and new comprehension. The model of Wilson, Shulman and Richert (1987) helped me to organize the tasks (i.e., analysis of two lesson plans and their teaching, and preparation and analysis of lesson plans on exponential function and video teaching episode).

At the end of this model of Wilson, Shulman and Richert (1987), I examined whether any change occurred on the participants’ perspective for teaching as a result of doing these tasks. In other words, I wanted to investigate whether or not participants improved their pedagogical content knowledge as a result of doing these tasks.

Using this model, I investigated the nature of pedagogical content knowledge of participants for teaching the concept of function. I looked at how the participants used different forms of representations, ideas, examples, activities and explanations in their lesson plan. When they taught their lesson on exponential functions and how they made it comprehensible to the group of high school seniors, I examined whether or not the participants recognized what made the learning of exponential functions easy or difficult for the group of high school seniors. I investigated whether or not the participants took into the consideration group of high school seniors’ current understanding to make instructional decisions. I looked at how the participants differentiated what the students comprehended and what they could perform (Graber, 1999), as well as how they provided the grounds for choices and actions. I also investigated the nature of relationships between the participants’ subject matter knowledge and pedagogical content knowledge for teaching the concept of function. I looked at how the participants used their subject matter knowledge of the concept of function for teaching.

Comprehension includes both substantive and syntactic structure. The substantive structures involve the ideas, facts, and concepts of the field, as well as the relationships among those ideas, facts, and concepts. The syntactic structures include knowledge of the ways in which the discipline creates and evaluates new knowledge. One should know what an exponential function is and how it is defined as well as how exponential functions are related to logarithmic functions. In addition, one should know different representations (equation, graph, table and verbal) to describe exponential functions and should be able to make transitions from one representation to another.
Transformation includes sub-processes: interpretation, representation, adaptation, and tailoring. Interpretation involves reviewing instructional materials from one’s understanding of the subject matter. Preparation includes investigating and critically interpreting the materials for instruction. This preparation process involves detecting and correcting errors in the text, and the processes of structuring and segmenting the material into forms. Adaptation of the materials helps teachers have a better and more suitable understanding of teaching. During this process, a teacher grasps and extends his/her repertoire (i.e., instructional strategies, programs, and conceptions). When a teacher examines the materials for instruction, he/she should possess a representational repertoire that consists of ideas (e.g., activities, illustrations, and examples) in a lesson (Shulman, 2004). Thinking through these key ideas helps a teacher consider the alternative ways of representing subject matter to students (Shulman, 2004; Wilson, Shulman & Richert, 1987). Adaptation includes fitting the represented materials to the characteristics of the students in general. One needs to consider students’ misconceptions or misunderstandings about the material (student’s ability, student level). Choosing appropriate materials is important because inappropriate materials may interfere with students’ learning of the concept. Tailoring refers to adapting materials to the specific student in one’s classroom instead of using materials for the student population in general.

In this study, the participants analyzed two lessons about probability that was given as assignment. They looked at how these lesson plans were prepared and taught. The participants were asked to compare the lesson plans and determine which one was better. During the analysis of the two lessons, the participants went through the transformation process, even though they did not prepare or teach these lessons. The participants had to look at the sub-processes of transformation to analyze these lessons. In addition, the participants prepared a lesson plan on exponential function for a small group of students. When the participants were asked to prepare a lesson, they used their subject matter knowledge and pedagogical content knowledge of the exponential functions. Then, the participants looked for different examples and activities for this lesson. They tried to choose the most appropriate examples and activities in their lesson plans and tried to adapt examples and activities that would be appropriate for the students’ needs.

Instruction refers to observable performance of the teacher in terms of effective teaching. It involves the most fundamental aspects of pedagogy, such as organizing the classroom, assigning and checking work, interacting with students, coordination of learning activities,
explanation, questioning, and discussion (Shulman, 2004; Wilson, Shulman & Richert, 1987). When the participants analyzed two lessons, they analyzed how these lessons were taught by an experienced mathematics teacher (Mr. A) as well as the performance of the teacher. In addition, the participants prepared a lesson plan on exponential functions and taught their lessons to a group of high school seniors. I videotaped their teaching to analyze their performance. I examined how the participants utilized their subject matter knowledge and pedagogical content knowledge of exponential functions to teach the lesson.

Evaluation occurs during instruction. The teacher checks both understanding and misunderstanding of students as he/she is teaching. Checking the understanding of students requires all forms of the participants’ comprehension and transformation. Evaluation is also directed at the teacher’s teaching methods and at the lesson. I utilized the videos of the participants’ teaching for evaluation. After their classroom teaching of the lessons on exponential functions, I conducted an interview with each participant. They watched the video clips and evaluated their teaching. They looked at what the group of high school seniors understood or misunderstood as a result of the teaching of exponential functions.

The teacher evaluates his/her instruction through reflection. Reflection is a process where by the teacher looks back at his/her teaching that has occurred. He/she recaptures the events and the accomplishments. In this study, reflection was done with the help of video teaching of the lesson plan on exponential functions. The participants watched video clips and then looked back at their teaching. They reviewed their forty-minute teaching from the beginning to the end of the lesson.

New comprehension refers to having an increased awareness of the purpose of new instruction. Having collected the data and conducted the interviews, I looked at whether or not the participants changed their perspective for teaching as a result of completing these tasks. I looked at whether or not the nature of their pedagogical content knowledge for teaching the concept of function changed as a result of completing these tasks.

Data Sets

In the following section, I will provide descriptions of these lesson plans, teaching of these lessons, the participants’ lesson plans on exponential functions, and their teaching of exponential functions as well as my observations.

I used these descriptions to analyze and compare the participants’ responses during the
interviews in this study. In addition, I wanted to provide some information for readers so that they would see what was done and how the data was analyzed in this study.

**Function Questionnaire**

The function questionnaire adapted from the studies of Even (1989) and Wilson (1992) included nineteen mathematics problems addressing different aspects of subject matter knowledge (e.g., examples of functions and non-functions, different ways of approaching of functions) and analysis of students’ incorrect solutions (see Appendix A). I did not intend to measure the performance and success of the participants with this questionnaire. Its purpose was to describe the nature of the participants’ pedagogical and content knowledge. The function questionnaire included different types of functions (e.g., linear, quadratic, exponential, logarithmic), definition of function and non-function, and different representation of functions (e.g., graphical, equation, set notation).

I asked the participants to show their method of solution and explain their answers so that I could examine the nature of their subject matter knowledge and pedagogical content knowledge of the concept of function. In their responses, I looked at how the participants classify relations into functions and non-functions. I examined their understanding of different types of functions. I looked at how the participants used different representations and transferred from one representation to another.

**Card Sorting**

The card sorting activity (Cooney, 1996) included twenty-eight different examples of seven types of functions (i.e., linear, quadratic, polynomial, exponential, logarithmic and trigonometric and rational functions) given in four different representations (i.e., tables, graphs, equations, and verbal descriptions) (see Appendix B). Each card was written on an index card and participants were asked to arrange index cards of functions into piles using both different representations and classification of functions. In the card sorting activity, the participants were asked seven questions and organized their knowledge about seven different categories of functions using different representations.

The card sorting activity helped me determine how the participants utilized different representations and how they moved from one representation to another. In addition, the card sorting activity was employed for assessing participants’ understanding of the relationships among different types of functions. I looked at how they identified the seven types of functions...
using four different representations.

**Analysis of Two Lesson Plans and Their Teaching Episodes**

At the time of the study, the participants, Jack and Sara, were enrolled in a MAE 4330 course in which prospective teachers’ analyzed middle and high school mathematics curricula from the perspective of adolescents’ learning of mathematics. They also discussed theoretical and practical aspects of learning and teaching mathematics in this class. Jack and Sara did not have any experience in writing a lesson plan so my major professor, the instructor of the MAE 4330 course, and I wanted the participants to gain some experience in preparing lesson plans as well as analyzing video teaching episodes of those lessons before they wrote a lesson plan and taught it for this study.

We asked Mr. A at Southern High School to participate in this study. Mr. A, had a bachelor degree and master degree in mathematics education and was pursuing his Ph.D. in mathematics education. Mr. A taught two years at the middle school level and one year at the high school level. He has been teaching undergraduate method courses for one year.

Mr. A agreed that he would prepare two lesson plans. The first lesson was on *Theoretical and Experimental Probability* (see Appendix C). Instead of preparing the lesson plan before implementing it, Mr. A taught the lesson unprepared, and then prepared the lesson plan. The second plan was on *Fundamental Counting Principles* (see Appendix D) and was written before he taught the lesson.

I videotaped Mr. A’s teaching of the lessons. I first gave the participants the lesson plans, then the videos of the teaching of those lesson plans. Later, I met with Mr. A to talk about the classes he taught for the study. I video-recorded the interview with Mr. A so that I could have the participants watch this video as well as describe his teaching from my point of view when I analyzed the data of the study. The prospective teachers in MAE 4330 watched the videos of the teaching of those lesson plans and discussed what they thought about Mr. A implementation of these two lessons. One sixty minute interview was conducted with each of the participants to discuss what they thought about the lesson plans as well as which lesson plan they found better and why.

**Two Lesson Plans**

**Summary of The First Lesson Plan**

The lesson plan called, *Theoretical and Experimental Probability* was created after the
lesson was taught (see Appendix C). The lesson plan was not designed prior to the lesson. The fifty-minute lesson was taught from the book and without any preplanned examples.

The purpose of this lesson was to have students differentiate between experimental and theoretical probability. For motivation, a real world example of finding probability would be presented to the students from a video of Rosa Parks. The students would be asked to work individually or work in groups throughout the lesson. The students with physical disabilities would be paired with students who could assist them. First, the students would be given the definition of experimental probability. Then, Mr. A would give the definition of theoretical probability. After the two definitions, the students would be presented with examples to find simple probabilities. They would be given red and yellow integer chips to compute and compare experimental and theoretical probabilities. If time permitted, the students would roll two dice and compute the sum of the numbers 100 times and compare with an addition table to compute theoretical probabilities. For assessment, the students would be given homework and would be graded. For accommodations, copies of notes would be handed out before or after the lesson to students who needed them.

**Summary of the Second Lesson Plan**

This fifty-minute lesson plan was prepared prior to the class. The purpose of the lesson was to have the students understand and use *Fundamental Counting Principles* (see Appendix D). For motivation, Mr. A planned to present a real world problem to the students. They would be asked to work individually or in pairs throughout the lesson. After working on the problem, the students reported their process for finding their responses as well as their results. A class discussion would occur concerning the validity of the responses and the procedures for finding the correct answers. After the discussion, the students would be led in the direction of using some organized manner by which to display their data using tree diagrams. They would be presented with real life problems in which they would have to utilize the *Fundamental Counting Principles* to find the correct answers. Through the discussion of completed problems, the students would see the multiplication aspect of the principle. Then, they would begin to see a need for a “shortcut” as stated by the *Fundamental Counting Principles*. At this point, Mr. A would present the standard way of relating the *Fundamental Counting Principles* to the students. A few problems that would be discussed and worked out earlier would be solved with the *Fundamental Counting Principles*. For the closure, the students would be asked to write their
own problem involving the *Fundamental Counting Principles* and to share these with class. The discussion would ensure the appropriate content and context of their problems that included the use of the *Fundamental Counting Principles*. If the time permitted, the students would write another problem. Then, they would trade their problem with a partner. The students would work on the problem and find the solution. They would discuss the appropriateness of each problem and the correctness of the answer. For assessment, Mr. A would assign homework that would be graded. For accommodations, copies of notes would be handed out before or after the lesson to students who needed them.

**Summary of the First Lesson Plan’s Teaching**

Mr. A taught these lessons to twenty-three high school students at the Southern High School. These students were seniors and had enrolled in liberal arts class, were for students who did not take Algebra 2. In this class, the students did not have quickness or ease of learning. Most of them had low reading levels. Mr. A described the students as low achieving. By the end of the semester, all twenty-three students had passed the class. Fifteen students received a grade of A; five students had a grade of B. Two students received a grade of C, and one student received a grade of D.

Mr. A began the lesson with the question, “what is probability?” He gave the students some time to think about the definition, and then called on a few students to give a definition of probability. They gave short answers such as “ratio,” “fraction,” “estimated,” “what I get,” “guess,” and “unknown.” After listening and taking notes about students’ responses at the board, Mr. A opened the textbook and told the students that they would look at two kinds of probability; *Theoretical and Experimental Probability*. He read the definitions from the textbook. He wrote

\[
P(E) = \frac{\text{# of outcomes}}{\text{total # of possible outcomes}}
\]

for probability and \(E = \text{event}\). Mr. A. said, “The book wants to say, an event is an instance we are interested in.” He explained the term *event* by giving some examples. For instance, he asked the students, “What is the possibility of Matt getting a seat in this class that had 13 chairs?” Then, he answered the question without allowing the students to answer the question. Mr. A wrote, “*Experimental Probability,*

\[
P(E) = \frac{\text{actual number of occurrences}}{\text{the possible outcomes}}.
\]

After reading the definition, he asked students to get into
groups of two and then gave each group one chip with one side yellow and the other side red. He asked them to find the probability of getting a red or yellow side when flipping the chip once. A few students said that the answer was two; others did not say anything. Mr. A told the students that I referred to the event and wrote \( \frac{1}{2} \) on the board. He said this was the correct answer because there were two colors on the chip.

Mr. A asked the students to work in groups of two. One student in each group would be the recorder and the other student would flip the chip. All the groups were asked to flip the chip 50 times and to record their results (i.e., the number of reds and yellows). After the groups worked this task for five minutes, each group came to the board and wrote their results on the ratio of yellow to red (see Figure 4.2).

<table>
<thead>
<tr>
<th>Group I</th>
<th>Group II</th>
<th>Group III</th>
</tr>
</thead>
<tbody>
<tr>
<td>23/27</td>
<td>26/24</td>
<td>28/22</td>
</tr>
</tbody>
</table>

Figure 4.2: Ratio of Yellow to Red

Mr. A told the students that the number of red and yellow should be related because experimental probability must be related to theoretical probability. He asked, “What can we do to get close to the theoretical probability of one half?” Then he gave the students graphing calculators and asked them to find the number of red and yellow for 500 trials using the calculator. He asked a student what he/she found and wrote the answer as a fraction on the board. He found this experimental probability close enough to \( \frac{1}{2} \). He told the students that the probability would get very close to theoretical probability (i.e., \( \frac{1}{2} \)) with more trials. He asked the students to make a table for fifty trials without flipping the chip and to write the probability of getting yellow or red. After they completed the task, Mr. A wrote the ratios of yellow to red on the board. He told the students that the number of yellow or red should be close to twenty-five. Toward the end of the class, he rolled a die on the smart board and asked the probability of getting 1, 2, and 7. After finding these probabilities, he gave the students homework from the textbook.

**Researcher’s Observation about Teaching of The First Lesson**

Mr. A did not prepare the lesson prior to his teaching. Ten minutes before the class, he looked in the textbook and found the definition of probability. There was not any logical order for how he presented the examples. He chose the examples right before he asked the students.
He gave the definition of theoretical and experimental probability from the textbook without discussion with the students. The definitions were given without taking into consideration what the students knew about probability. Throughout the lesson, Mr. A told students what to find and what the result would be. The students were not given an opportunity to explore possible outcomes of experimental probability and find the relationship between the theoretical and experimental probability. When they were asked to do the activities, they were not given an opportunity to evaluate their findings. Mr. A gave the students all the results and explanations. He utilized the chip activity, asking students to find the probability of the number of red and yellow and write their findings. Then, he explained what needed be learned from the activities such as theoretical probability, experimental probability and the relationship between the two concepts.

**The Summary of Teaching of the Second Lesson**

Mr. A began the lesson with the following problem (see Figure 4.3) and asked the students to work on the problem in groups of two.

Mr. A went to Publix to get a sub. He could choose white or wheat bread, turkey, ham, or roast beef meat, Swiss, American, or Provolone cheese, and a small or large Coke. How many choices of meals can he have?

![Figure 4.3: Daily problem](#)

The students worked on the problem for a couple of minutes. As they worked on the problem, Mr. A walked around and checked their work. Then, he wrote the students’ name in each group on the board. While he was walking around, he asked, “What are my choices? How many different options do I have? How will I not get the same meal?” in order to have them explain their findings. After the students worked on the problem for five minutes, he wrote their results on the board and asked each group to explain how they found their answer.

He asked them to scale the correctness of their answers from 1 to 10 to find out how sure they were about their answer. When the students were explaining their reasoning and solution strategies, Mr. A took short notes on each group and their solution strategies. During this discussion, one group changed their answer. Mr. A asked this group to explain their answer at the board. The group utilized a tree diagram to find all the different options. Mr. A told the class what this method was called and how they could use this diagram to solve problems about the *Fundamental Counting Principles.*
Mr. A gave another example involving the *Fundamental Counting Principles*. He asked one of the students to list the characteristics of a woman he would go out with. The student described different characteristics: good or bad outfit, dress size (8, 9, or 10), G.P.A (A, B, or C). In order to find out how many different women a man could go out with, Mr. A showed all the possible outcomes using a three-diagram on the board. Then, he asked them if there was a better and easier way to find the solution without using a tree diagram.

In groups, the students worked on the example for a couple of minutes and found all possible outcomes; then they shared and discussed their findings as a class. The students found the answer by listing all the outcomes. They did not think of multiplying the number of options in each group. At this point Mr. A showed them how to solve this problem using the *Fundamental Counting Principles* as well as the solution with a tree diagram so that the students could see that both methods would give the same result.

Mr. A wanted to go back and solve the first example (see Figure 4.3) using the *Fundamental Counting Principles*. He asked the students to find the number of options in each group (bread, cheese, meat, and drink) and obtain the answer.

After solving the problem, he stated the definition of the *Fundamental Counting Principles*, gave another example (see Figure 4.4) on the smart board, and asked students to work on it for a few minutes.

A school has 3 math teachers, 4 English teachers, and 2 Spanish teachers. How many different sets of teachers could a student have for these 3 subjects?

Figure 4.4: Example 2

After solving the example, the students were asked to explain what they found. With the help of the students, Mr. A solved the second example using both *Fundamental Counting Principles* and a tree diagram on the board.

Mr. A gave another example (see Figure 4.5) for the students and asked them to solve it using a tree diagram or the *Fundamental Counting Principle*.

Earlena is dressing up for Halloween. She can choose from 2 wigs (red or blond), 2 fake noses (bulbous or pointy) and 2 pairs of glasses (green or mirrored). How many different disguises can she create, and what are they?

Figure 4.5: Example 3
The students worked in groups to solve the problem and then shared and discussed their findings as a class. He wrote the solution on the board with the help of the students.

After this example, the students were asked to write a question involving the **Fundamental Counting Principles**. The students worked individually on this task. After they finished writing a problem, the students shared their problems with their classmates. After sharing the problems, each student read his/her question and solved it verbally in the class.

**Researcher’s Observation about Teaching of The Second Lesson**

The lesson was planned prior to teaching. When Mr. A planned this lesson, he thought how he would explain to someone who had absolutely no idea about probability. Mr. A chose examples for this lesson because he wanted to break the concept into small units and then build it into whole. Mr. A stated with a word problem that involved the **Fundamental Counting Principles**. With these examples, Mr. A was able to make transition from one example to another. With creating these examples, he made sure the students understood fundamental counting principles.

In the teaching of this lesson, the students were involved in solving all the examples. In these examples, the students saw the place of mathematics in their life. Mr. A made the examples applicable to students. The examples that involved a female student’s different possibilities for Halloween costume and woman’s characteristics received the whole class attention. With these examples, the students made connection between mathematics and real life. In these examples, Mr. A asked students to scale the correctness of their answers from 1 to 10 to find out how sure they were about their answers. By asking this question, Mr. A’s goal was to help students realize their own uncertainty so that they would evaluate their answers. He did not want to tell the groups that their answer was not correct or incorrect. As Mr. A expected, the students worked on the question and changed their answers. Throughout the teaching of this lesson, the students shared and discussed their findings with the class.

**A Lesson Plan on Exponential Function**

In this study, the participants did not have any experience in writing a lesson plan so my major professor, the instructor of the MAE 4330 course, and I decided to give a lesson plan guideline including objectives to the participants.

This way, the participants prepared the lesson. Their preparation focused on goals and procedures of the lesson. The decision to give the objectives of the lesson was supported by my
committee members as well. Without objectives, the committee suggested that the participants prepare a lesson that might be related to exponential function, but it would not be designed for teaching exponential functions. In addition, the committee suggested that the participants should be kept in the same direction so that I could compare their lessons.

**Teaching Episode**

With the analysis of lesson plans and teaching videos of these lessons, the participants gained some experience into determining which of these lessons were prepared and taught better. Then, the participants were given a lesson plan guideline with objectives and asked to write a lesson plan on exponential functions. The purpose of having participants write a lesson was to investigate how they used their subject matter knowledge and pedagogical content knowledge for preparing a lesson plan and teaching. I examined the nature of the relationships between the participants’ subject matter knowledge and pedagogical content knowledge for teaching exponential functions.

I video-recorded their teaching so that I could investigate the nature of pedagogical content knowledge of participants for teaching the exponential functions. I had the participants watch video clips of their teaching evaluation and reflect on their implementation of the lessons.

During the interview with each participant, I asked them what they thought about their teaching and whether any change occurred in their thinking about how they should teach exponential functions. This enabled me to examine how the participants transformed their subject matter knowledge for teaching.

**THE CASE OF JACK**

Jack is a 21-year-old university senior who has been attending the Southern State University and majoring in mathematics education. The secondary mathematics and middle grades mathematics teacher certification programs are approved by the State Department of Education and are credited by the National Council for Accreditation of Teacher Education. A student preparing to teach secondary school mathematics have to take at least twenty-one (21) semester hours of mathematics, statistics, and/or computers beyond the common degree prerequisites. For Jack, mathematics has shown him various focuses throughout his life. His earliest memories of mathematics involved practical application with finances (e.g., learning
about prices, comparison of various prices). Jack has been interested in learning more advanced and abstract concepts, which held his focus when he learned about negative numbers in the second grade. Mathematics was not his favorite subject until he took pre-algebra and algebra. In his autobiography, Jack wrote,

The teacher I had for those two years had such an intense desire to teach the material and that enthusiasm was passed down to me. Math, at this point in time, became less of a routine algorithm and more a practical way to interpret various aspects of the world. The precision and notation of concepts instantly drew me into the field.

Jack’s high school mathematics background involved being accepted to the International Baccalaureate Organization (IBO) program in high school, which enabled him to attend several honors classes such as algebra, geometry, trigonometry, pre-calculus. Jack truly enjoys algebra and its concepts in mathematics because his mathematics teachers had the skill and eagerness for teaching high school students. “The material became less of a chore and more of a game, thanks in large part to the instructors and the material itself. This was the point when I began to consider the mathematics education degree in college.”

Although Jack speaks most fondly of his mathematics teachers in high school, he speaks less affectionately of his college courses. His university education had involved taking calculus I, calculus II, linear algebra, introduction to applied statistics, and elements of geometry courses. In these college courses, Jack found that little time was spent on the explanation of the concept, compared to his previous high school courses.

Jack has a 3.58 overall college G.P.A. He plans to teach high school, but he is getting both middle and high school certificates. He is interested in teaching algebra, geometry and pre-calculus courses. Jack is a bright, articulate, friendly and talkative person who is eager to be a mathematics teacher. During the interview he said,

I enjoyed math. It is kind of persuaded knowledge in a way like the general eagerness to continue learning. When I looked at into education, I look specifically at math because of my interest in the subject but also I just thought that teaching would be a good fit for my interest.

Jack also completed mathematics education courses: these were teaching high school mathematics, using technology in teaching mathematics and using history in teaching mathematics. However, at the time of the study, he was enrolled in three methods courses (teaching mathematics in middle school, problem solving in teaching mathematics and how
adolescence learn mathematics) and two mathematics courses (college geometry and elements of algebra).

Jack has always been good at mathematics and enjoys it very much. During the interview, he said, “math is always one of those subjects that I enjoyed and I think other people [Pause] could enjoy in right environment.” Jack thought that the methods are used in mathematics classes are not designed towards students’ learning. He believes that methods that are used in mathematics classes should be varied. He said, “I find that the projector method worked when the teacher stood up there and explained the concept that works for me. I know that, but it doesn’t work for everyone.” However, he strongly believes that “the teacher is kind of an authority on the matter, but not necessarily [Pause] the person where all the information has to come from.”

Jack’s formal teaching experience was in his method course (teaching high school mathematics). In this course, he observed middle and high school classes and taught three days lessons in a row. At the time of the study, he was tutoring middle and high school students on the university campus as a service to the community.

**Function Questionnaire**

**Question 1**

In the first question, Jack was asked to give an example of a mathematical function. He drew a curve and wrote an algebraic equation $f(x) = x$ as an example (see Figure 4.6). Then, for the examples, he wrote, “Passes vertical line test, for every $x$ only one $y$.”

![Figure 4.6: Jack’s example of a Function](image)

**Question 2**

For the second question, Jack was asked to give a non-example of a function. Jack drew a parabola opening to the right (see Figure 4.7) as an example of non-function.
He was unable to write an equation, which is not a function. Again he utilized the vertical line test to determine that the graph (see Figure 4.7) was not a function. The following excerpt illustrates his reliance on the vertical line test.

Jack: I just drew a side way parabola that passes the vertical line test. I did not write a function for that one. I figured that would represent non-function.

Interviewer: What would be the equation for this graph?

Jack: [Pause] that was what I was trying to come up with earlier. \( f(x) = -x^2 \) I don’t know. I think there is. If this vertex was on actual axes [long Pause] I haven’t seen it so long. I don’t know. I am not sure whether the negative goes outside or inside.

Interviewer: What happens when it is inside and outside?

Jack: If it is inside, it would always be positive because inside get squared. I guess it would have to be outside [long Pause] I think it just flips it upside down. I can’t think of the equation for that. I don’t know.

As can be seen in the transcript above, Jack failed to write the equation of a parabola opening to the right. He first thought that \(-x^2\) would open to the right, then said, “I think it just flips it upside down. I can’t think of the equation for that.” He did not know what would create two y values for each x value in the domain. In other words, he did not think of writing \( y^2 \) to have a relation, which is not a function.

In questions 3-7, I asked Jack to identify whether or not the relations represent a mathematical function (see Figure 4.8).

| Identify whether or not each of the following represents a mathematical function. |
|-----------------------------------------|----------------|
| 3. \( 2x - 3y = 5 \)                  | 6. \( \{(1,2),(2,3),(3,4),(4,5)\} \) |
| 4. \( x^2 - y = 5 \)                  | 7. \( \{(1,2),(-2,4),(3,4),(1,1)\} \) |
| 5. \( x^2 + y^2 = 4 \)                |               |

Figure 4.8: Identifying mathematical functions
Question 3

For question 3, Jack wrote the equation in a slope-intercept form to determine whether or not it was a function. After writing the equation, \( y = \frac{2}{3}x - \frac{5}{3} \), he knew that it was a line and all lines (except for the vertical ones) would pass vertical line tests. I noticed that his understanding and use of vertical line test were very limited. He was trying to find out what the graph of the equation (i.e., \( 2x - 3y = 5 \)) looked like so that he could use the vertical line test.

Jack: I wrote in slope intercept form. From that I was able to figure out that line. So it would pass the vertical line test. There is a relationship for functions where every \( x \) relates to one \( y \).

Interviewer: How do you know this is going to pass the vertical line test?

Jack: Because it’s the line. Because this is \( y \) intercept that equation is line. That’s how I knew a line would pass the vertical line test. I mean as long as it’s not a vertical line.

Question 4

Jack solved \( x^2 - y = 5 \) in question 4 for \( y \) and wrote \( y = x^2 - 5 \). At this point, he knew that it was a parabola and would pass the vertical line test. He again preferred to use the vertical line test on the graph of \( x^2 - y = 5 \). He did not attempt to solve the equation algebraically or numerically to show that \( x \) values cannot have more than one \( y \) value.

Jack: That one, I put it into \( y \) equals form \([ y = x^2 - 5 \]). I knew that was a parabola. I knew that it would pass the vertical line test. That’s a normal and regular parabola. It is not skinner or fatter. It was moved over a certain amount, which would be “5”. “5” represents how far it’s shifted to the left or right or to the bottom. I can’t remember.

Question 5

In question 5, Jack immediately recognized that \( x^2 + y^2 = 4 \) is a circle and said, “Circle cannot be a function.” The following excerpt indicates his tendency to visualize the graphs of equations.

I knew it was a circle. That equation represents a circle. I knew circle can’t be function because it can’t pass the vertical line test. I pretty much look at the graphs in my brain. I am a fairly visual thinker.

Question 6-7
Questions 6 and 7 are given in set notation. I wanted to investigate how Jack would interpret different representations of functions and non-functions. Jack looked for ordered-pairs, which had the same y values and easily determined the function and non-function.

Jack: For that one [Question 6], I looked, there is 1, 2, 3, 4 represented for first variable and 2, 3, 4, 5 are for the second variable. So I knew that there was no overlapping and there wasn’t any between the two. 1 goes to 2, 2 goes to 3, 3 goes to 4 and 4 goes to 5. [Pause] There wasn’t any four of second variable going 2 and 3 first. Whereas number 7 that’s kind of the case where 4 went to both -2 at the first variable and 3 at the first variable. So that [Question 6] represents function. Whereas number 6 it is.

**Question 8-9-10**

In question 8, 9 and 10 (see Figure 4.9), three graphs were given and the participant was asked to determine whether or not they were functions given in graphical form. These two questions were very easy for Jack. He said, they (i.e., questions 8 and 10) were functions because the graphs would pass the vertical line test. Question 9 was not a function since a circle was not a function. It would not pass the vertical line test.

![Figure 4.9: Question 8-9-10](image)

By asking questions 3 though 10, I wanted to examine Jack’s solution strategies as well as his interpretation of different representations of functions and non-functions. Questions 3, 4, and 5 are given in algebraic representation, questions 6 and 7 in set notation, and questions 8, 9 and 10 in graphical representation. I observed that when he was given relations in algebraic representation, he tried to determine what their graphs looked like in order to use the vertical line test. He did not think of solving the relations algebraically or numerically. For example, he could have substituted values for x or y in the equation $x^2 + y^2 = 4$ to show that some x values could have two y values. He also used the vertical line test for the questions 8, 9 and 10.
However, he did not explain why the graphs were or were not functions or did not mention the univalent property of functions (Even, 1990) in his definition. For example, in question 8, he drew some vertical lines on the graph and said, “that is a circle. It doesn’t pass the vertical line test.” However, he failed to explain what it meant when a vertical line would intersect the graph at two points. When I asked Jack how he would respond if a student said question 5 and question 9 were functions, he talked about comparing the graphs and equations but did not think of using a different approach or tried to explain why they were not functions.

Jack: [Long Pause] Graph the equation and find the equation for the graph. Kind of show them similarities between each set of representations. What is the difference between these two? Equations are fairly similar in the way they look how is one function and not a function. Have them question that. Possibility with the graph, have them look at it. Ask them why this is a function and not a function. In this equation right here, it was a function when you graph it. It looks a lot like this, not a function. Have them explore and discuss that possibly.

**Question 11**

Jack was asked to determine the relationship between the length of a side of a square and its area. He first wrote $S_1, S_2, S_3,$ and $S_4$ on the sides of the square and $S_1 = S_2 = S_3 = S_4$. Then, he said,

Length of a side of a square and its area. We know that area of the square or the area of any rectangle would be the length times width. Because it’s a square and the length is equal to the width, the area is any side of the square times itself. We know that relationship between the length of a side of a square and its area. Area is equal to the sides of the square.

He correctly wrote the equation, $Area = S \cdot S = S^2$ that represents the relationship between the length of a side of a square and its area. When I asked him if this equation was a function, he drew the following graph of $A = S^2$ (see Figure 4.10) to determine if it was a function.

![Figure 4.10: Jack’s graph for Question 11](image)
The following excerpt clearly indicates his tendency to use graphical representations instead of algebraic or numerical approaches.

Because I guess you can represent it as an equation where the area is equal to side of the square. It is basically a different way of writing your x’s and y’s. That equation \( A = S^2 \) right there, if you had graphed it, you would have the area for this [y] axes, side for that [x] axes. The result would be something like that. In geometry, you can’t really have negative sign. So this part of the graph [second quadrant] doesn’t really apply.

**Question 12**

In the first part of question 12, Jack was asked to give a definition of a function. He wrote, “An equation where for every x value you have one and only one resulting value.”

During the interview, Jack thought he revised his definition of a function, but his definition was still narrow.

Jack: An equation where for every x value you have one and only one resulting value. I guess I should probably revise that because this one is not necessarily an equation. I don’t know, this entire process kind of refined my definition of a function. It is just a little different exercise. I basically say data in the function is basically the way of representing the relationship between the two sets of data. I guess you can look at it as two variables, two sets of data.

For the second part of question 12, Jack was asked to give an alternate version of a definition of a function, if a student does not understand his first definition. He wrote, “Graphically a function is when it passes the vertical line test.” He drew a Venn diagram to represent a function and numerical values in the domain and range (see Figure 4.11). He also said that showing how x and y correlate graphically might be helpful when students had difficulty understanding the data or the equation. The following excerpt illustrates his thinking,

[Long Pause] I guess I put a function graphically, when it passes the vertical line test. I guess that would be helpful. If the student doesn’t necessarily understand the data, you can also represent it as an equation. Show them how x’s and y’s correlate graphically. Generally that could help quite a bit because students can visualize and see themselves the vertical lines. That’s how it represents x values and that sort of thing. They would be able to see visually. It’s another way for them to see, have representation of functions, I guess. Graphically that would be a good alternative, if the students do not understand the broad definition.

As can be seen in the transcripts above, Jack was aware of the importance of univalence nature of functions (Even, 1990). However, he assumed that function was a relation between two sets of data (numerical data) when he was defining a function. He did not refer to the
arbitrary nature of the two sets. That is, “functions do not have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers” (Even, 1990, p. 96).

During the interview, Jack mentioned how he was taught a definition of function and drew the following Venn diagram. This shows that Jack was aware of representing a function with a Venn diagram but did not prefer to use it to introduce the definition of a function.

![Figure 4.11: Jack’s Venn Diagram](image)

**Question 13**

Jack was asked to find the real solutions of a quadratic equation (see Figure 4.12). The equation, $ax^2 + bx + c$ is a quadratic function where $a$, $b$ and $c$ were real numbers with $a \neq 0$. The parabola would open upwards if $a > 0$ or downward if $a < 0$. If “a” is positive or negative, when $x=1$, it has positive value (y value) above the x-axis. So there is a real solution on x-axis. When $x=6$, it has negative value (y value). It goes below x-axis. It has to go above x-axis. Therefore, there would be two real solutions.

If you substitute 1 for $x$ in $ax^2 + bx + c$ (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

![Figure 4.12: Question 13](image)

Jack’s answer on the paper, “Infinite amount of solution because there will be a given range and you have an infinite amount of numbers in that range” indicates that he failed to distinguish the range and the real solutions of the quadratic equation. In other words, he was assuming that the equation, $ax^2 + bx + c = 0$, would have infinitely many solutions because there were infinitely many numbers in the range even though a quadratic function could have a maximum two real solutions. Although Jack was familiar with equations and graphs of quadratic functions since high school, he did not attempt to make the connection between the two representations, equation and graph of the quadratic or manipulate the inequalities $a + b + c > 0$ and $36a + 6b + c < 0$. He only wrote inequalities to find the real solutions of the quadratic equation. Jack did not try to look at another representation of the question (Even, 1989).
Question 14

A student is asked to give an example of a graph of a function that passes through points A and B. The student gives the following answer (See figure). When asked if there is another answer the student says: “No.” If you think the student is right- explain why. If you think the student is wrong-how many functions?

For question 14, Jack stated that there were infinitely many functions that went thorough the points A and B. However, he did not refer to the arbitrary nature of the two sets on which the function is defined (Even, 1989). In his response, Jack showed non-linear functions and passed through two points (i.e., A and B).

There could be [Pause] it could go all the way up and all the way down or around. There are so many different ways that they can create. Those two points could be a simple parabola right there. There are so many x’s and y goes on infinitely. So I mean there could be the same curve, but a thousand out.

When he was asked the second part of question 14 (see Figure 4.13), he suggested giving the same question to students and sharing different graphs with other students. I believe having students work on the problem and eliciting students’ work would be a very effective way of approaching this problem.

One good way of doing that would be to show the answers the other students have and show how there is such a big variation. I figure if you give the same problem to different students, they probably will come up with a lot of them. They would come up with straight lines. But if they’ve seen parabolas before, if it has been a topic that they’ve seen before, I am sure some of them will draw parabolas and different things. Kind of show them how different is here and different is there [see Figure 4.14]. I mean it is still function. It still passes through these points, but I think showing them a lot of different examples would kind of force the idea that there are infinite variety. There is no way of
writing or demonstrating graph of different varieties. I think showing wide range of examples.

Figure 4.14: Jack’s solution to Question 14

**Question 15**

In order to examine his knowledge of relationship between functions and equations, I asked Jack to explain how functions and equations were related (Question 15). He said, “All functions can be represented as equations, or you know basically functions are subset of equations. Not all equations are functions but all functions can be represented as equations.” (see Figure 4.15).

Figure 4.15: Jack’s response to Question 15

Then, to justify his answer, Jack gave the following example:

I guess if we look at it graphically, two different functions, I mean two different graphs, both can be represented as equations [see Figure 4.16]. So we have \( y = x^2 \) and \( y = (-x)^2 \). I think second is negative x square, no [Pause] yes. Something along this line, it will be an equation, represented as an equation, but this [first graph] would be a function passes the vertical line test. This [see Figure 4.16b] won’t be a function, if it is not passing the vertical line test. While both can be represented as equation, [Pause] the only one would be a function. I don’t think there is an example of function… the one that I drew on the other page [see Figure 4.14]. I can’t really define it as an equation itself but it would still be a function. I mean if there is a possibility of defining as an equation. Like I am not exactly sure how or what.
Jack believed that all functions could be represented by an equation. This shows his limited understanding of functions. He did not think of the arbitrary nature of functions (Even, 1989). That is, arbitrary objects that cannot be described by an equation are considered as a function. He assumed that functions must have equations even though he was aware of different representations of functions. Jack said, “Equations are a way of representing functions, one way of out of several different ways of representing functions. Not to say every given equation is a function.”

Question 16

By asking question 16 (see Figure 4.17), my goal was to examine how he would use the shape of the graph to determine the signs of the coefficients “a”, “b”, and “c” or vice versa. The relationship between the signs of the coefficients is very simple. “a” is negative (i.e., $a < 0$) if and only if the parabola opens downward. The x coordinate of the vertex is $-\frac{b}{2a}$ and the y coordinates which is the function value $f\left(-\frac{b}{2a}\right)$. Since “a” is negative, the function value of y is at maximum (i.e., positive). “b” and “c” are positive if and only if the value of y coordinate of the function value (i.e., $-\frac{b}{2a}$) is positive, another way Jack could have used a derivative to determine the sign of “b”. The first derivative of this function would give $f'(x) = 2ax + b$.

When the equation (i.e., $2ax + b = 0$) is solved for x, the x value (i.e., $-\frac{b}{2a}$) would be the vertex of the parabola. Since the parabola opens downward, “a” has to be negative (i.e., $a < 0$). The x coordinate of the vertex (i.e., $-\frac{b}{2a}$) can be used to determine the sign of “b”. Since x coordinate of the vertex is positive and “a” is negative, “b” has to be positive to have a positive x coordinate.
This is the graph of the function \( f(x) = ax^2 + bx + c \). State whether \( a, b \) and \( c \) are positive, negative or zero. Explain your decision.

Figure 4.17: Question 16

Jack was not sure whether “a” and “b” were negative or positive so he wrote, “Either “a” or “b” must be negative to have an upside down parabola.” He also stated that “c” was positive because the vertex was higher than the x-axis”. During the interview, I asked him whether he could determine the signs of a or b:

What I would probably do is to use a, b, and c to, solve the equation for multiple data. I haven’t seen quadratic equations for many years. I don’t know the short cuts in that. If “a” is positive or negative what exactly means.

Jack could not remember how to determine the signs of “a” and “b” using the relationships between the graph and the coefficients of a quadratic equation. He correctly determined the sign of “c”. However, his reasoning was not correct. Jack thought that when zero was substituted for x, “c” would be the only value and the vertex (or the highest point) of the parabola. Even though the value of “c” determines only the y-intercept, Jack thought that y intercept (i.e., the value of “c”) was the vertex of the parabola. Therefore, c was positive. In addition, he did not think of using the formula, \(-\frac{b}{2a}\) for the x coordinate of the vertex to find the sign of “b”.

Interviewer: How do you know c is positive?
Jack: Because if x is zero, [Pause] this entire term right there \([ax^2]\) would be zero, so would this term \([bx]\). “c” would be the only thing left. So I figure if the y value is equal to the “c”, that would probably be the highest.

Question 17

In question 17 (see Figure 4.18), Jack was asked to interpret a student’s answer.
A student was asked to find the equation of a line that goes through A and the origin O.

She said: “Well I can use the line y=x as a reference line. The slope of line AO should be about twice the slope of the line y=x, which is 1. So the slope of the line AO is about 2, and the equation is about y=2x, let’s say y=1.9x. What do you think the student had in mind? Is she right? Explain.

Jack wrote, “Accurate estimation, but she didn’t find a definite equation.” on his paper. During the interview, to explain why the student could not find the equation, he said:

I guess it is depending on the scale. If we are looking at it direct one-to-one instead of being positively point 5 to 1 or you know something similar, [Pause] he could use $y = x$ as a reference (see Figure 4.19).

Estimation can serve as a description of a student’s work. However, this general definition is not satisfactory. The student should be encouraged to calculate the exact value of
slope by comparing the y value by the x value (Even, 1989). Jack knew that student’s the estimation was not correct and he suggested using \( y = x \) as a reference. At this point, I wanted to know how Jack could guide the student in order to find out his approach this problem as a teacher,

The student is using what she understands about slope and everything to come to the conclusion. What I would do is to I either have her [Pause] depending if this is exactly what is given, [Pause] not anything like take labels or mark anything what I would have her do is to take out a ruler to define. That’s a good estimation but what I would have her do is have her measure, find out what x is. Just inches, some given unit. Like it doesn’t matter what exact unit is as long as the unit right here. She measures that. It is the same unit. She measures by the same unit, and she is able to find rise over run and then find the slope and be able to find pretty accurate line for that.

Jack’s approach was appropriate and student-centered. Instead of teaching how to solve the question or saying that the student’s answer was incorrect, he preferred to have the student create scales and measure x and y values to find the slope. I believe he had a good understanding of the problem and this enabled him to come up with a strategy that could help the student estimate the slope better. However, he did not say specifically what the correct answer was, how he was going to help the student, and what the student needed to do.

**Question 18**

The graph of \( y = \frac{1}{x^2 - 1} \) is not continuous at 1 and -1. The limits where x approaches 1 from left and right, \( \lim_{x \to 1^+} f(x) = \infty \) and \( \lim_{x \to 1^-} f(x) = -\infty \), as well as limits where x approaches -1 from left and right, \( \lim_{x \to -1^+} f(x) = \infty \) and \( \lim_{x \to -1^-} f(x) = -\infty \) should be used to draw the graph of this function. Substituting values for and plotting those points on the coordinate plane does not give the graph of the function (Even, 1989).

Jack was asked to explain to a student in algebra 2 how to sketch the graph of \( y = \frac{1}{x^2 - 1} \). Jack suggested two methods on the paper, which could be called the point-wise approach and use of asymptotes (undefined points). He wrote,

1. One simple method to demonstrate would be to make a table of values their resulting values from the resulting graph, an investigation into how the exact equation can be graphed quicker would be beneficial for the student. (2) Another possible method for graphing would be to factor the terms in the denominator and to thus realize that this graph contains two asymptotes; this method would perhaps be easier for the student if
certain concepts (mainly asymptotes have been previously discussed in class). Both are appropriate methods and might demonstrate a student’s level of understanding on their certain abilities.

During the interview, Jack thought a combination of these two methods would be appropriate to explain how to sketch the graph of \( y = \frac{1}{x^2 - 1} \).

I probably use kind of a mixture what I would do I would factor the bottom two terms \([ (x-1)(x+1)]\) and then, I would recognize the lines where asymptotes. Then, I would probably pick a couple of points and then graph my equation based on how these points look.

Jack knew that the function is undefined at 1 and -1. However, he could not remember how to draw these asymptotes (i.e., \( x = 1 \) and \( x = -1 \)) or whether these lines were vertical or horizontal.

Here is the coordinate system [see Figure 4.20]. We do not want zero to be in that dominator when either of these terms right here is zero, then we know that they are all being asymptotes right there because you can’t divide by zero. So when \( x \) is equal to 1 or \( x \) is equal to negative 1 then \( y \) [Long Pause] \( y \) is undefined. That’s where an asymptote occurs and at that point. [Long Pause] I can’t remember whether or not they’re vertical lines.

![Figure 4.20: Jack’s graph for question 18](image)

Even though Jack took advanced mathematics classes up to Calculus I and II, he should have used limits to determine how the function behaved when \( x \) was approaching 1, -1, infinity, and negative infinity. Graphing and teaching of how to graph functions cannot be based on substituting values for \( x \) and plotting those points, or using asymptotes.

**Question 19**

The exponential functions (i.e., \( y = a^x, a \neq 1 \) and \( a > 0 \)) are the inverse of logarithmic functions (i.e., \( y = \log_a x, a > 0, a \neq 1, \) and \( x > 0 \)), and are not used very often. The following procedure shows how to find the inverse function of \( f(x) = 10^x \) (see Figure 4.21)
In question 19, Jack was asked to determine whether or not log function and root function were inverse of $f(x) = 10^x$. Jack wrote on the paper,

$$y = 10^x \quad \text{Interchange variables}$$

$$x = 10^y$$

$$\log x = \log 10^y$$

$$\log x = y \log 10$$

$$y = f^{-1}(x) = \log x$$

Figure 4.21: Solution of Question 19

The student is correct in that there exist two different methods of defining the inverse. However, because of the definition of inverses (and in this case the ability to show the equality of the two inverse equations), the two equations must be equivalent. Thus, for all purposes, these two equations are both the same function.

Jack interchanged the variables in the equation, $y = 10^x$, then took the logarithm of both sides and wrote $\log(x) = y$. To find the second inverse function, he wrote $x = 10^y$ and took the yth root of both sides but realized that the equation was not the same as $\log x = y$. At this point, he was still thinking that he must have made a mistake somewhere because taking the logarithm and yth root of both sides should have given the same function. The following excerpt clearly shows that Jack over-generalized the idea of a root function. For example, $y = x^2$, $x \geq 0$, is the [inverse of square root of x (i.e., $\sqrt[2]{x}$)] inverse of yth root of x or the function $x^n$, $x \geq 0$, is the inverse of nth root of x (i.e., $\sqrt[n]{x}$). Jack assumed that there was no difference between taking the log of both sides and taking the yth (or xth) root of both sides. Jack thought finding yth root of x equals to ten (i.e., $x^{\left(\frac{1}{y}\right)} = 10$) would give him x value. He ended up having the value of 10.

Even though Jack could not find the x value by taking y square root of both sides, he still taught that this algorithm should give the inverse of $f(x) = 10^x$. Jack thought that log and root functions were two different ways of writing inverse of $f(x) = 10^x$ (see Figure 4.22).

I was able to find one inverse, and the other one, I think I messed up somewhere and I wasn’t able to find the second one. I know just by the definition of inverses that pretty much for any equations. If it has an inverse, then the inverse would only go back to the equation. In order for that equation to have two inverses, those two inverses have to equal each other. [Long Pause] I guess ways of representing it through equations, the inverse, but two ways of representing are equivalent so, I mean there’s just one function because there’s just one [Pause] there’s two sets of data. It’s the same sets of data that go
into the function. But this is the two ways of representing the same data or the relationship between the two sets of data that are representing two different equations.

Figure 4.22: Jack’s solution for Question 19

Summary of Function Questionnaire

In this section, I summarized Jack’s responses and work in the function questionnaire to describe his subject matter knowledge for teaching functions under six aspects (Even, 1989): *essential features, different representations, the strength of the concept, basic repertoire, alternative ways of approaching, and different kinds of knowledge and understanding of function and mathematics*. In addition, I utilized an additional aspect, *analysis of students’ mistakes*, to describe Jack’s pedagogical content knowledge of the concept of function.

**Essential features**

This aspect refers to the correspondence between the arbitrary nature and univalence of functions. Today, univalence property is accepted as an important part of the definition of concept of function. When the univalence property is not explained well by a teacher, mathematics might look like an arbitrary collection of rules and definitions. In other words, once the students were introduced to the vertical line test, they ignore the concept and apply the rule (Fernandez, 2005). In the case of Jack, he was familiar with univalence property and its use as a criterion for telling whether a relation was a function. He thought it was important for students to know about the requirement of having only one value in the range for each element in the domain.
In questions 1 and 2, Jack gave an example of function and non-function. In question 1, he drew a curve and wrote an equation for an example of a function. Jack utilized the vertical line test to determine how both the graph and equation represent a mathematical function. In question 2, Jack drew a parabola opening to the right, and then tried to write the equation of the parabola. Jack thought the equation would be \(-x^2\) instead of \(y^2 = x\) or \(y = \sqrt[3]{x}\). He did not realize that this equation represented a function. He was unable to write the equation opening to the right. Again, he utilized the vertical line test to show the graph did not represent a mathematical function.

In questions 3 through 10, relations were given in different representations (e.g., graph, equation, set notation). Jack tried to find out what the graph of the equation looked like so that he could use the vertical line test to determine whether the relations was a function or not.

In question 12, Jack defined a function as a relationship between two sets of data. As an alternative definition of a function, he said, “graphically a function is when it passes the vertical line test.” His definition of a function as well as his responses in the interviews revealed that he did not refer to the arbitrary nature of functions. The interview also revealed Jack’s excessive use of the vertical line test without explaining what it meant to fail the test, why it worked or why it was or was not a function.

**Different representations of functions**

This aspect is about recognition of the same idea in different representations, manipulation of the idea within a given representation, and translation of the idea from one representation to another (Even, 1989). It is clear from the literature (e.g., Lesh, Post, & Behr, 1987; Janvier, 1987; NCTM, 2000) that having multiple ways –for example, visual and analytic – to represent mathematical concepts is beneficial. Even (1989, p. 127) says, “flexibility in moving from one representation to another allows one to see rich relationships, to develop a better conceptual understanding and strengthen the ability to solve problems.”

The question 11 is about the relationship between the length of a side of a square and its area. Jack successfully translated the question written verbally into algebraic form and represented the relationship using an equation \((\text{Area} = S \cdot S = S^2)\). In question 13, Jack was asked to find the real solutions of quadratic equation. He thought that there would be infinitely many solutions because there were infinitely many x values and corresponding y values. He was unable to solve the problem algebraically. He did not think of approaching the problem
graphically. For example, a quadratic equation can have two solutions at most depending on the value of the discriminant. If you approach the problem graphically, a parabola can have maximum two x intercepts.

In question 16, Jack was given a parabola and asked to determine whether or not “a”, “b”, and “c” were positive, negative, or zero. The quadratic function is one of the fundamental functions in the high school curriculum and prospective teachers have been taught and have used this function during their high school and college education. At this level, they should be able to use different representations of quadratic functions. Jack failed to make a connection between graphic and algebraic representations (or translate from one representation to another) and was not able to find the signs of the coefficients. He took advanced mathematics classes up to calculus II, but he did not think of utilizing derivation (i.e., the maximum of the parabola) to determine the signs of “a” and “b”.

To give an example of non-function, Jack drew a parabola opening to the right for question 2. He failed to write the equation of the parabola opening to the right. Jack thought the equation would be \(-x^2\) instead of \(y^2 = x\) or \(y = \sqrt{x}\).

In question 18, Jack was given the equation of a rational function (i.e., \(y = \frac{1}{x^2 - 1}\)) and asked to draw its graph. When he took the function questionnaire, he wrote that he would make a table (i.e., plotting points) and found the asymptotes to draw the graph of this function. However, he was unable to draw the graph or translate the function from algebraic to graphical form during the interview.

Overall, Jack did not seem to have a good understanding of different representations of functions (e.g., quadratic, rational). He had difficulty making a connection between representations (e.g., graphical and equation) as well as translating one representation to another (e.g., equation to graphical).

**The strength of the concept**

This aspect is related to teachers’ understanding of important characteristics of the concept. Even (1990) states that understanding of the concept of function must involve an understanding of the structure of functions and inverse functions. The question 19 was about inverse functions. For example, during the interview after the function questionnaire, Jack was asked to determine whether or not \(f(x) = 10^x\) had two different inverse functions. He found the...
inverse of the exponential function by taking the logarithm of both sides. However, he overgeneralized the idea of root function and thought that taking yth root of both sides would give the same inverse function (or logarithm function). Jack knew how to take the logarithm of both sides and was able to find the inverse function using his procedural knowledge. However, he did not know the relationship between a function and its inverse in an algebraic representation or graphical representation. Taking the yth root of both sides to find the inverse function clearly indicated that his algebraic and graphical understanding of exponential, logarithmic, and root functions was very weak and based on algebraic rules.

**Basic Repertoire**

This aspect includes powerful examples of basic functions such as linear, quadratic, polynomial, exponential, logarithmic, and rational. Even (1990) says that “the basic repertoire should be well known and familiar in order to be readily available.

Jack was given linear functions in algebraic and graphical forms in questions 3 and 8, respectively, and quadratic functions in algebraic and graphical form in questions 4 and 10. He recognized the functions and determined whether or not they were functions.

In question 17, he analyzed the student’s incorrect solution about the slope of a linear function. He knew that the second line (OA) was steeper and greater than the slope of $y = x$, so he said that the slope was not correct and could be estimated by measuring the x and y coordinates of the point A. However, he did not say how the student should choose the scale or the length of one unit. This clearly indicated that Jack had a good idea to help the students but failed to come up with a specific solution strategy to solve the problem.

In question 13, when Jack was asked to solve a quadratic equation, he failed to use fundamental properties of quadratic functions. He did not think of using the discriminant or approaching the problem graphically to find the maximum number of solutions. Similarly, he could not use the relationship between the quadratic equation and its graph in question 16. Jack was able to recognize the equations and graphs of the quadratic functions in the questionnaire but his understanding was not deep and conceptual.

When Jack was given the question 19 about inverse functions, he failed to utilize the general knowledge about exponential and logarithmic functions. He was unable to use some basic theorems, properties or graphs to solve the problem. Since he could not solve the problem algebraically, he could have approached the problem graphically and shown the logarithmic and
exponential functions were inverses of each other or showed that the root function was not the inverse of exponential function. Similarly, for question 18, he could have utilized a graph of a similar rational function to draw the graph of \( y = \frac{1}{x^2 - 1} \).

In question 14, when Jack was explaining why there were infinitely many functions going through points A and B, he was mentioning a parabola or a curve. In other words, he was thinking of smooth and continuous functions. He could have drawn the graph of important functions from the high school curriculum (linear, quadratic, polynomial, exponential, logarithmic, and rational) as well as piece-wise or discontinuous functions. His misconception about functions surfaced during the interview when he was answering the question 15. He said, “All functions can be represented as equations. Basically functions are subset of equations. Not all equations are functions but all functions can be represented as equations.”

A teacher should have a good understanding of these basic functions as well as a repertoire of these basic functions. This also enables them to come up with alternative ways of representing examples to students. Otherwise, a teacher might have difficulty finding an appropriate example to teach a concept or to solve a problem. Jack was able to recognize the graphs or equations of the functions. However, he had difficulty approaching the problems using alternative ways due to his limited representational repertoire as well as weak subject matter knowledge.

**Alternative ways of approaching**

This aspect includes different ways of approaching functions (e.g., point-wise, interval-wise, global). Even though he has taken advanced mathematics classes, Jack failed to utilize limits or global approach to draw the graph of \( y = \frac{1}{x^2 - 1} \) in question 18. He should have determined the behavior of the function as \( x \) approaches \( \pm \infty \) and \( \mp 1 \). When he took the function questionnaire, he wrote that he would make a table (i.e., point-wise approach) and then he would find the asymptotes. However, he could not draw the graph in the interview.

**Different kinds of knowledge and understanding of function and mathematics**

The sixth aspect consists of conceptual and procedural knowledge of functions as well as meanings and understanding. I observed that Jack’s understanding of functions was procedural and rule oriented. For example, when he was trying to find the inverse of the exponential function (see Question 19 of Function Questionnaire), he was only relying on his procedural
knowledge. As a result, he thought that taking the logarithm and yth root of both sides would give the same inverse function. Since he could not remember the theorem for inverse functions or the algebraic rule for logarithm and root, he failed to find his mistake. In addition, since his procedural understanding was not supported by another representation (e.g., graphical), he was unable to approach the problem using a different approach.

In questions 13 and 16, he was unable to solve the problem about quadratic equations because he failed to remember the rule. This illustrates that his understanding of quadratic functions was not deep and conceptual. In addition, his understanding of the relationships among different representation of quadratic equations (e.g., graphical, algebraic) was not strong enough to lead him to the correct answer.

In the following aspect, Jack’s pedagogical content knowledge is described. This aspect is closely related to the ones discussed above (Even, 1989).

**Analyses of student’s mistakes**

When Jack was analyzing the student’s incorrect estimation of the slope of a line in question 17, he knew that the slope was incorrect because the second line was steeper and had a greater slope than the slope of $y = x$. He suggested that the slope could be estimated by measuring the x and y coordinates of point A. He said he would have the student measure the x and y coordinates and find the rise over run (see Figure 4.17). This illustrates that his approach to this problem would be student-centered because he wanted to engage the student instead of solving the problem. However, he did not say how the student should choose the scale or the length of one unit. This clearly indicated that Jack had a good idea of how to help the students but failed to come up with a specific solution strategy to solve the problem due to his limited pedagogical content knowledge. If Jack had strong pedagogical content knowledge or known more about common misconceptions about students’ mistakes, he could have suggested a more appropriate strategy to help the student.

In question 14, in order to show that there were different functions going through points A and B, he said he would give the same questions to other students and share their different functions with the whole class. This is a good example of his student-centered pedagogical knowledge. However, in order to help the student create an appropriate image of a function, he should have used piece-wise and discontinuous functions as well as smooth and continuous basic functions (e.g., lines, parabolas). I believe this excerpt (see question 14 in the Function
Questionnaire) indicates his difficulties recognizing sources of students’ mistakes as well as weak pedagogical content knowledge about students’ misconceptions of functions.

For the first part of question 12, Jack wrote the definition of a function: “An equation where for every “x” value you have one and only one resulting value” when he was asked to give an alternative definition to help the student understand a definition of a function. When he took the function questionnaire, he wrote, “Graphically, a function is when it passes the vertical line test.” During the interview, he also said that it would be helpful to show that the function passes the vertical line test. He also drew a Venn diagram to show how he was taught the definition of a function. Jack was aware of representing a function with a Venn diagram but did not prefer to use it to introduce the definition of a function. During the interview, it became clear that Jack was not taking the student’s possible difficulties or misconceptions about functions into account when he was suggesting these examples. For example, giving an example of non-function can be very helpful. If you want to use the vertical line test, you should explain why the test works and what the purpose is. One should not use the vertical line test as an explanation of what a function is.

Jack drew a Venn diagram representing a one-to-one function (see Figure 4.11). As a teacher, I would give several examples of functions, constant functions, one-to-one functions, and non-functions using Venn diagrams in order to help students create correct image of a function. A teacher should know or anticipate sources of students’ common mistakes and take them into account when one makes instructional decisions. Otherwise, students may have a different understanding of a function if you give typical examples of functions and use the vertical line test as an explanation of what a function is.
**Card Sorting Activity**

**Question 1**

Consider cards 1, 12 and 27. What is the easiest way to sort these cards into two piles? What criterion would be used?

<table>
<thead>
<tr>
<th>1</th>
<th>12</th>
<th>27</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Graph 1" /></td>
<td><img src="image2.png" alt="Graph 12" /></td>
<td><img src="image3.png" alt="Graph 27" /></td>
</tr>
</tbody>
</table>

**Figure 4.22: Question 1**

Function cards 1 and 12 are given in graphical representations. The function on the card 27 is written verbally. The graph of the function on card 1 is exponential. The function on the cards 12 and 27 are periodic.

In question 1, Jack was asked to sort function cards 1, 12 and 27 into two piles (see Figure 4.22). Jack wrote functions for cards 1 and 12 and dotted graph for function card 27. From his answer, it seemed that he put cards 1 and 12 in the same group and separate the function card 27. During the interview, he stated that function cards 1 and 12 were both graphs but that function card 27 was a situation or word problem. It was obvious that he classified the function cards according to their representations. In other words, Jack considered function cards 1 and 12 as graphical representations and function card 27 verbal representation.

Jack: 1, 12, 27. When I looked at the cards the easiest way of separating them was into two piles. 1 and 12 they, it was functions or yes, it was functions and their representations 1 and 12 were both graphs. 27 was a situation word problem. It was just real easy to see based on a representation one begin graph one being actual like words. I mean two being graphed, on the words.

When Jack was asked whether or not these three cards could be sort differently. He told me that he could not think of any other way to sort these cards.

**Question 2**

If you were to sort 28 cards into four piles, which criterion makes the sorting easiest?

**Figure 4.23: Question 2**
Twenty-eight function cards could be sorted into four piles using different representations; equation, table, verbal, and graph. Function cards 7, 9, 10, 13, 15 and 18 are given as an equation. Function on the cards 4, 14, 16, 21, 25 and 28 were given numerically in table representations. Function cards 3, 5, 8, 11, 23, 24, 27 are given in verbal representations. Function cards 1, 6, 12, 17, 19, 22 and 26 were given in graphical representations.

Jack was asked to sort 28 cards into four piles (see Figure 4.23). He wrote “linear, trigonometric, exponential, miscellaneous” on the paper. In the interview, he said that it would be easier to sort the cards according to representations (i.e., graph, equation, verbal, and table).

Jack: Looking back, I figure four criteria that would be easier for piles to have would be the different representations the graphs, the tables, the situations word problems and the equations would be the easy ones. Instead I just kind of looked at how the data the relationship between the data and I had the linear relationships and exponential relationships, and trigonometric and the miscellaneous category, because I can’t exactly fit the rest of them into this one, categories of miscellaneous. Looking back, different representations probably would be a better way of breaking into four piles.

Interviewer: Why did you choose these three, then linear and exponential, and trigonometric?
Jack: Because I think probably at the time I wasn’t exactly looking at it as the function as their different representations, like I would still recognize one of the tables, one of the graphs, you know an equation, but I think I was looking at it as far as kind of what the graph looks like, what’s the relationship between the data. When I recognize the representations I didn’t really at that time see as a criterion for sorting.

The excerpt indicates that Jack was able to recognize some of the functions (i.e., linear, trigonometric, exponential) and put the other functions in the miscellaneous category because he did not recognize all the functions given the activity and it was difficult for him to organize them according to their types.
Question 3

Sort cards 3, 15, and 19 into two piles. In what sense is your sorting criterion different from the criterion used in exercises 1 and 2?

3.
Denise is filling a cubical container measuring one foot on each edge with water. She notices that it takes a lot more water when each dimension of the cube is increased. She wonders how much the volume of the cube increases when each dimension is increased x units.

15. \( y = 2x + \pi \)

19.

![Figure 4.24: Question 3](image)

In question 3, Jack was asked to group function cards 3, 15 and 19 into two piles (see Figure 4.24). Function cards 15 and 19 are linear functions given in equation and graphical forms respectively. Function card 3 is a polynomial function given in a verbal representation. A cubical container measuring one foot on each edge with water has a volume of 1. When each dimension is increased x units, each dimension will be \( 1 + x \) and the volume will be \( (1 + x)^3 \).

Each dimension is 1
The volume of the cubical container is \( (1)^3 = 1 \)
Each dimension is \( 1 + x \)
The volume of the cubical container is \( (1 + x)^3 \)
The increase in volume = \( (1 + x)^3 - 1 = x^3 + 3x^2 + 3x \)
where \( x \) is the increase on each side.

![Figure 4.25: Solution of Card 3](image)

The increase in the volume of a cubical container is equal to \( x^3 + 3x^2 + 3x \) (see Figure 4.25). This increase in volume can be denoted by a polynomial function. Jack thought that function card 2 represented an exponential function and function cards 15 and 19 represented linear functions. As a result, he grouped function cards 15 and 19 together and separated function card 3.

Jack: I think I put number 3 as exponential, 15 and 19 as linear.
Interviewer: How did you determine these were exponential and linear?
Jack: I looked at it, and I tried to visualize it and in my brain. She notices that it takes a lot more water when each dimension of cube increases. So, if you are graphing as far as when [Pause] it’s if you are looking at it as a graph of at this dimension how much water does it take it? You look at the progression as far as that goes. I figure it would probably be exponential. For 15 and 19, the equation looked linear as far as that goes and for 19 it also looks like a line. So, kind of where I looked at it.

I wanted to investigate how Jack would calculate the increase in the volume and why Jack thought that card 3 was an exponential function.

Interviewer: How did you determine it was an exponential function?
Jack: That’s a good question. My gut reaction is probably be exponential. Sort of relationship between two variables. It’s hard for me to visualize without having like doing it myself in a way. Like I know what she is doing, she is filling a basically cubes. There’s just increasing dimension on it [see Figure 4.19]. This is x, this is $x + 1$. So all the sides would be $x + 1$ in length as far as the dimension goes. As far as it goes, there would be $x^3$ for looking at the volume. $(x + 1)^3$ as far as volume goes. Just length times width times height. That’s one, $1^3$. I guess it depends on what you are being asked to graph.

![Figure 4.26: Jack’s solution of the function card 3](image)

Even though the dimension of a container was 1 and each dimension was increased by $x$ unit, Jack thought that each dimension was $x$ initially and each dimension was increased 1 unit (see Figure 4.26). As a result, he thought the volume of the container was $x^3$ and the volume of the second container was $(x + 1)^3$. He drew the graph of these two functions on the same coordinate plane, and then substituted 1 and 2 for $x$ in the equation $y = x^3$ to show the increase in the volume of the container. For some reason, he drew the graph of $x^3$ steeper than the graph of $(x + 1)^3$. He didn’t attempt to find the difference between the volumes. He should have found
the difference between the volumes of $1^3$ and $(x + 1)^3$, and then drawn the graph of volume of a cubic container versus dimension. He tried to interpret the difference in the volume numerically. He did not think that the difference could be written as a function of $x$.

**Question 4**

In what sense are cards 6, 8, and 21 alike?

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>Fred is considering which size pizza is a better buy. He wonders what happens to the area of the circular pizza when the diameter of the pizza is doubled.</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>y</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 4.27: Question 4

Function card 6 is a parabola given in a graphical representation. Function card 8 is a quadratic function presented verbally. Function card 21 is also a quadratic function given numerically in a table. Jack was asked to determine how these cards were alike (see Figure 4.27).

In his response, Jack wrote that function cards 6 and 21 were exponential. For all there cards, he wrote, “all shows possible results for an equation.” During the interview, Jack changed his answer. First, he found the area of the circular pizza when the diameter of the pizza was doubled. He determined that the question on the function card 8 represented a quadratic function (see Figure 4.28)

Figure 4.28: Jack’s solution of the Function Card 8

Jack: I put all shows possible results for an equation. I don’t know. I don’t think I really understood exactly the way they are or they are all like. Again, exponential was the
wrong way of saying that. I guess for number 5, if they are talking about the area of a circular pizza is $\pi r^2$, they are asking when the diameter of the pizza is doubled. In a sense, we know that diameter is twice the radius and so of the diameter doubled. It would be 4 times $r$. In order to get that we solve the equation for “$r$” half a “$d$”, area

$[ A = \pi \left( \frac{d}{2} \right)^2 ]$ this kind of the equation for number 5. They’ll be using it to represent the function. They all seem to be quadratic functions of some sort. I mean it could be a substitute, could have x there and y right there, but the way sound of is quadratic function. So I guess they’re all quadratic functions.

Function card 6 was given in graphical representation. From the graph, Jack immediately determined that it was a quadratic function. For function card 21, Jack looked at the table values. From x and y values, he decided that function card 21 was also a quadratic function (see Figure 4.29).

![Figure 4.29: Jack’s graph of Card 21](image)

Interviewer: How do you know these are quadratic functions?
Jack: Because I call tell pretty much because the x is just going of one unit, so we know that’s uniform. But the y looks like it goes from 1 to 0 and back up to 1 then 4 and then 9 I guess it would produce in either direction even further. You get more data. You would know. Probably you would follow the same sort of [Pause] so I mean they are both, if you graph all three. They would all look like similar functions that are quadratic.

Interviewer: How do you know “6” is quadratic?
Jack: That’s a good question. I just looked at it and my brain kind of screams quadratic. I don’t know. I think it’s like impulse response. It’s like this is a graph of what the equation would look like. What kind of graph is this? I mean I can pull points from it. Represent it that way. Maybe I can even figure out what the equation is and look from the equation. Kind of how represent it. Graphically it looks like quadratic.

As the excerpt above indicates, Jack was able to recognize three different representations of quadratic functions (i.e., graph, verbal and table) and determine how those functions were alike.
Question 5

Sort cards 9, 13, 17, and 22 into different piles, two different ways. Place the cards into the following 2 x 2 matrix; C₁ and C₂ represent one type of criterion and D₁ and D₂ represent a different type of criterion.

<table>
<thead>
<tr>
<th></th>
<th>C₁</th>
<th>C₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>D₁</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D₂</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9
\[ y = 3x^5 - x^4 + 4x^3 - 10x^2 + x - 6 \]

13
\[ y = \frac{2}{3x} \]

17

22

Figure 4.30: Question 5

In question 6, Jack was asked to sort cards 9, 13, 17, and 22 into different piles and two different ways (see Figure 4.30). The function on card 9 is an equation of a polynomial function. The function card 13 is an equation of a rational function. The functions on cards 17 and 22 are given in a graphical representation. The function on card 17 is a rational function. The function on card 22 is a trigonometric function. In this question, Jack was asked to sort these four cards into two piles using two different ways.

Jack sorted the functions according to their representations. He indicated that the functions on cards 9 and 13 were in the same group because together they were equations. He grouped the functions on cards 17 and 22 because they were graphical representations.

9 and 13 are both equations. 17 and 22 are both graphs. So I put it across like that. I think D₂, 13, 17 grouping and 9 and 22 grouping. I looked at it as I knew that both of these have asymptotes, 9 and 22 are just both strictly continuous.

The second criterion that Jack used was continuity. He stated that the graphs on cards 9 and 22 were continuous and the functions on cards 13 and 17 had asymptotes (see Figure 4.31). Jack utilized a correct criterion and sorted the functions successfully.

Interview: How do you know these two have asymptotes?
Jack: Because I know that I guess the x=0 one right there would be an asymptote. Right here, whenever x is zero on the bottom, we know that y would be undefined. I guess they [13, 17] both have x=0 asymptote lines.

9,13 17,22
13, 17 9,22

Figure 4.31: Jack’s solution of Question 5

Question 6

Sort cards 1, 6, 12, 16, 17, 21, 25, and 28 into two piles. Describe carefully the criterion you used to sort the cards.

Figure 4.32: Question 6

Function cards 1, 6, 12 and 17 are given in graphical representations. Functions on cards 16, 21, 25 and 28 are given numerically in a table.

Jack was asked to sort function cards 1, 6, 12, 16, 17, 21, 25, and 28 into two piles (see Figure 4.32). Jack put them into two groups using representations. He grouped the function on cards 1, 6, 12 and 17 using graphical representation. He put function on cards 16, 21, 25 and 28 in the same group because they were given in tables.

I sort them into two representations. One was graph. One was table representation. For graphs, 1, 6, 12, and 17, they are all graphical representations of functions. 16, 25, 28, 21, they were all tables. That’s how I grouped them into a way to represent it.

Then I asked Jack whether or not these cards could be sorted differently.

I think as far as having to do in two piles, two distinct piles, there are different ways of classifying them but as far as sticking all those into two piles. That’s probably the only way I can see it.

Jack did not recognize the types of the functions in the question and could not sort them according to their types.

Question 7

Sort cards 1, 6, 7, 16, 21, 26 and 28 into two piles. Describe carefully the criterion you used to sort the cards.

Figure 4.33: Question 7

Jack was asked to sort function cards 1, 6, 7, 16, 21, 26 and 28 into three piles using two different criteria (see Figure 4.33). The functions on cards 1, 6 and 26 are given in graphical representations. The function on cards 21, 16 and 28 are given numerically in a table. Function
cards 7, 18 and 20 are equations. Function card 21 can be denoted by the equation \( y = (x + 1)^2 \). The function on card 16 can be represented by the equation \( y = \log x \). The function on card 28 can be written as \( y = 2^{-x} \). Function cards 6, 7 and 21 are quadratic. The functions on cards 1, 18 and 28 are exponential. The function on cards 16, 20 and 26 are logarithmic functions. These functions can be sorted into three groups according to their representations (i.e., graph, equation, and table) as well as according to their types (i.e., exponential, quadratic, and logarithmic).

In his response, Jack grouped the function cards into three piles according to their representations. He grouped the function on cards 1, 6 and 26 because they were given in graphical representations. Jack put the functions on cards 21, 16 and 28 together because they were given as tables. He indicated that the functions on cards 7, 18 and 20 were in the same group because together they were equations.

I think I broke it down to representations the same way. Graphs 1, 6, 26, and equations, 7, 18, and 20, and tables 21, 16 and 28. 1, 6 and 26 they are all graphs, and 7, 18, 20 they are all functions represented by the equation form and then 21, 16, 28 they were all tables.

During the interview, Jack was asked whether or not he could sort the function cards using a second criterion. Jack found the second criterion, which were the categories of exponential, quadratic and logarithmic functions.

Interviewer: Can you sort these cards differently?
Jack: [Long Pause] that might be possible since there are three groups getting into. Exponential, maybe logarithmic, and quadratic.

He checked each function card to make sure he was correct. At first, Jack thought that the functions on cards 6 (quadratic), 26 (logarithmic), and 18 (exponential) were exponential. Then, Jack seemed to use the representations of function cards 6 (quadratic) and 7 (quadratic) to classify them as a quadratic function. The function on card 6 was given in graphical representation. The function on card 7 was an equation.

Interviewer: How do you know?
Jack: I mean I looked at different representations and none of them looked specifically linear, none of them looked specifically periodic. “6”, “26” this one right here looks like exponential function. “18”, this right here is an exponential function, the equation [Pause] “6” looks more like a parabola. “7” looks similar to “6”.

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Jack’s hesitation about grouping these function cards continued as he talked about the other functions on the cards. Again, he changed his answer. He first said that function cards 7 (quadratic) and 21 (quadratic) were quadratic, and then card 7 (quadratic) could be exponential. For the function on card 20 (logarithmic), he was not sure whether it was exponential or logarithmic.

Jack: 20 might be exponential but I’m not exactly sure but it should have its own group for logarithmic function. “21” looks quadratic. “7” looks another quadratic, possibly. I’m not sure [Long Pause] “7” is probably form of representation of exponential functions. I’m not positive about that though. That [7] doesn’t look like it. That’s [7] probably quadratic.

Jack thought the function card 16 (logarithmic) was exponential. He also said, the function on cards 20 (logarithmic), 26 (logarithmic), 28 (exponential) were similar and could be logarithmic. He plotted the points and tried to match the points to the graph of a familiar function (i.e., logarithmic, exponential, or quadratic) instead of trying to find the pattern in the tables. As a result, he said he needed more data (or points) to determine the shape of the graph.

Jack: It’s [16] only a certain side of data. I don’t know what goes beyond if you’re looking at the x’s and the negative region or beyond 100. I mean it looks like it will probably be something exponential, but I’m not positive.

Interviewer: Do you think we need more points to determine what kind of function it is? Jack: I think the more data you have the more you can be sure. I am not positive though. I mean it [card 20 logarithmic] looks really similar to what “26” looks like. I’m thinking 20 and 26 are. This [28] might be logarithmic function but I don’t remember what the graph looks like specifically.

Jack decided to draw the graph on the card 16 by plotting the points given in the table to determine what it looked like (see Figure 4.34). Jack correctly determined that the function on card 16 was logarithmic, however, he denoted the logarithmic function using an exponential equation \( y = 3^x \) instead of writing \( y = \log_{10} x \). Then Jack put the function on card 16 under the category of logarithmic function. He said that logarithmic and exponential functions were inverses of each other but did not remember how to write the equation of the logarithmic function whose graph he drew. As can be seen in his work, the graph intersected the y-axis and did not cross the x-axis at \( x = 1 \) even though the logarithmic function \( y = \log_{10} x \) did not cross or touch the y-axis and crossed the x-axis at \( x = 1 \). Jack knew that the function should be one-to-one to have an inverse function. However, he did not remember how to find the inverse of a
logarithmic function and claimed that he would interchange x and y to find the inverse function. When I asked him to explain what he did;

    Jack: [Pause] I know that let’s say $y = 3^x$. Logarithmic function is similar to that. Log is basically a way of function. If switch x and y’s, trying to find inverse [Long Pause]. I know they [logarithmic and exponential] are related in the way that I think [Drawing the graph of the function on the card 28 [see Figure 4.35]]. They are sort of inverses of one another, but I can’t specifically say how [Pointing the following graphs, [see Figure 4.34 and 4.35]]. It has been a while since I worked with them. Basically switching x’s and y’s are basically, that’s what the inverse is. I guess the only way to switch all the x’s and y’s [Pause] the way for it to be a function, would have to be for every x, there is only one y. For every y there is only one x. There is only way for both function and its inverse to be function.

Figure 4.34: Jack’s graph of the Function Card 16

Figure 4.35: Jack’s graph of the Function Card 28
Jack also attempted to draw the graph of the function on card 28 (see Figure 4.35) by plotting the points given in the table. He first thought that the equation of the function was $y = 2^x$, and then he changed the equation and wrote $y = 2^{-x}$. Toward the end of the interview, I asked him to put the functions into three groups (see Figure 4.36).

<table>
<thead>
<tr>
<th>Exponential</th>
<th>1, 18, 28</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadratic</td>
<td>6, 7, 21</td>
</tr>
<tr>
<td>Logarithmic</td>
<td>16, 26, 20</td>
</tr>
</tbody>
</table>

Figure 4.36: Jack’s solution of the question 7

Even though Jack did not have a good understanding of exponential and logarithmic functions, he determined their types of the functions by plotting the points given in the tables and correctly sorted them into three groups according to their types. During the interview, it became clear that Jack’s strategy was mainly point-wise for the functions represented by a table. When he drew the graph of a logarithmic function card 16 ($y = \log x$), he failed to write the correct equation for the graph. In addition, he made some mistakes on the graph of $y = \log_{10} x$ which showed deficiencies in his content knowledge about logarithmic functions.

**Summary of card sorting activity**

In question 1, Jack did not recognize function cards 1, 12 and 27, but he was able to sort them using their representations (graph and verbal).

In question 2, I asked Jack to sort 28 cards into four piles. Even though he did not recognize some functions in the activity, he sorted the cards using their representations (graph, verbal, equation and table).

In question 3, Jack recognized the linear function given as equation (card 15) and in graphical representation (card 19). However, he made a mistake, failing to represent the volume as a function of $x$. He assumed that each dimension was increased 1 unit instead of $x$ and tried to compare and graph $x^3$ and $(x+1)^3$. He drew the graph of both functions and substituted values for $x$ to show the increase in the volume.

In question 4, three quadratic functions (cards 6, 8 and 21) were given in graphical, verbal and table representations. He was asked to determine how these cards were alike. He successfully translated card 6 from verbal to equation form. Jack also recognized the quadratic
relations in the table on card 21. So, he knew that all these functions were quadratic. At this point, it became clear to me that Jack only understood quadratic functions well. He was sorting the other functions by only using their representations.

In question 5, Jack was given four functions (cards 9, 13, 17, and 22) and asked to sort them using two different types of criteria. He sorted the function cards according to their representations (equation and graph). As a second criterion, he used continuity. Jack said 13 and 17 had asymptotes, 9 and 22 continuous. Therefore, they should be grouped separately.

In question 6, Jack again sorted the function cards according to their representations. He could not sort them according to their types.

In question 7, Jack sorted the functions cards according to their types (i.e., exponential, quadratic, logarithmic). When I asked him to sort them using different criteria, he tried to sort them according to their types. Even though Jack had difficulty determining exponential and logarithmic functions and he used an exponential function to represent a logarithmic function, he sorted them correctly according to their types. His strategy of determining the types of functions relied on plotting points and matching them to familiar functions. Jack also made some mistakes on the graph of \( y = \log_{10} x \) which showed his deficiencies in his subject matter knowledge. Overall, he sorted the function cards according to their representations. His understanding of basic functions (quadratic, linear, exponential) was weak and limited. He seemed to understand different representations of quadratic functions. However, in the function questionnaire Jack was unable to make connections among different representations (e.g., graph, algebraic) to determine the signs of “a”, “b”, “c” or to solve the quadratic equation. Similarly, in the card sorting activity, I noticed that Jack was able to recognize exponential and logarithmic functions given as equation and graph, but he had difficulty determining exponential and logarithmic forms given in tables. In the function questionnaire as well as in the card-sorting activity, Jack failed to find the inverses. He was not sure about how to find inverse of logarithmic and exponential functions. In the function questionnaire, he took yth root of both sides. In the card-sorting activity, Jack said he would interchange x and y in the equation to find the inverse of a logarithmic or exponential function. At this point, it became clear to me that Jack’s understanding of logarithmic and exponential function was very weak. He could recognize the logarithmic and exponential functions if they were given in other forms. In
addition, his understanding of the relationships among these representations was fragile and limited. As a result, different representations of these functions were separate and not connected.

**Analysis of two lesson plans and their teaching**

**Jack’s analysis of the two lesson plans and their teaching**

Jack analyzed two lesson plans and then analyzed videos of the teaching of these lessons. The description of these lesson plans as well as their teaching videos are provided at the beginning of this chapter.

The first lesson (or lesson plan) focused on *Theoretical and Experimental Probability* (see Appendix C) and the second lesson (or lesson plan) focused on *Fundamental Counting Principles* (see Appendix D). Jack was asked to determine which of these lesson plans he liked better and to explain his reasoning. The following excerpt illustrates his thinking:

I definitely like the second one [fundamental counting principle] better in the way that material was presented. I know that the second one, it wasn’t just stating definition. In the second day, I believe it was more kind of discovery type learning. Experience [Pause] I thought is fundamentally more towards [Pause] I guess better for the students like their understanding of the concepts then being able to kind of discover the information. Overall I thought it was a fairly good lesson in the way that it was formulated toward to students [help] develop their own understanding. I think that’s probably not nearly as time efficient as giving them information and having them work with the information, but I think this is a lot more advantages for the students’ understanding. Therefore, if time permits or it’s a section that important enough to take time then, this sort of method I think is quite helpful.

Jack thought the second lesson was designed in a way where the students (high school seniors) examined the concept through discovery learning. He correctly determined that the students were presented with real life problems where they investigated *Fundamental Counting Principle*. When I asked Jack why he did not like the first lesson:

In the first one, it was kind of general by the book definition represented to the class and then work’s done. Well, it was first day [theoretical and experimental probability], I did not like giving the definitions, like here with lesson procedures the standards will be given with the definition of experimental probability and it will be given the theoretical probability. He [Mr. A] did go into asking them kind of what they thought probability was. He asked them what is the probability. As soon as the students were able to get some part of it, I remember this one, if you answer specifically, took something along the line. He was getting it how it’s kind of expected outcome and overall outcomes. He was kind of hinting it at that little bit. So, I mean he did try to get to students develop a definition for themselves but he [Pause] at the same time the lesson didn’t really a [Pause] kind of drive at having students learn. It’s kind of giving them the definition in the formula for, so I thought just giving it isn’t necessarily the most advantageous method
for teaching that certain lesson. I thought the second one was better in a way that it’s situated the way the information is given or discovered.

As can be seen from the excerpt above, Jack did not like the idea of giving the definitions and formulas for *Theoretical and Experimental Probability* from the textbook at the beginning of the first lesson (*bad*). In the first lesson, Mr. A asked students to give a definition to get them to think about what probability was. Then, he gave the definition of probability before the students discussed the definition or learned the activity.

I asked Jack whether or not Mr. A achieved the goals for this lesson. He was not sure whether or not the teachers achieved his goals for the purpose of this lesson. He was not sure whether most of the students understood *Theoretical and Experimental Probability* and were able to distinguish between these two definitions. The following excerpt explains his thinking:

I think if they were listening to some of the discussion that took place in sort of thing, they’d probably be able to kind of make the comparison between the theoretical and experimental. They would probably understand that but [Pause] no, there is the possibility that not all of them were listening or paying attention.

I asked Jack to compare these two lessons (*bad & good*). For the first lesson (*bad*), he knew that giving definitions or information to the students was not helpful for their learning. In the second lesson (*good*), he thought that the students were given an opportunity to examine *Fundamental Counting Principle*. The following excerpt explains his comparison of these lessons.

So, just regurgitating it, and telling the students and lecturing the students on the information isn’t a necessarily helpful for them to learn the information. But by leaving them in certain direction, do certain things into learn about it, that’s fundamentally more important then just making sure they have a chance to hear the information whereas this is where they are actually processing it and [Pause] interacting more of it in a way [Long Pause] it says in the procedure [the second lesson] that the students will begin to see a need for and want a short cut in the fundamental counting principle. In that respect, I think the lesson plan was driving at having students doing a long hand version and then kind of from their own hand version being able to [Pause] possibly come up with a sort cut. That’s it. By that, they’re kind of falling into same path that mathematicians used to find the fundamental counting principles back in the day and from that they were able to kind of follow the same mind of reasoning as the mathematicians. So, their understanding of the concept would be better then telling the students this is fundamental counting principles. It’s relevant because of this and it’s useful and used certain problems. So, you work a couple, you have them work problems in a certain situation.
From that being able to come up with this principle. I think is quite bit more helpful to students’ understanding.

I also asked Jack how he thought the questions helped and guided the students. He observed that the first lesson (bad) focused on testing students based on the given rules and definitions. However, in the second lesson (good), the students were guided to examine definitions and rules through the examples.

Jack: [Pause] I think in the second lesson plan specifically the questions weren’t so much what you think of this. It was more. The questions were there to guide the students to come to an understanding about the concept for the fundamental counting principles. It was guiding them towards end of it also [Pause] guiding them to kind of refine it. [The Second Lesson Plan] says here how they’re finding their language in order to come up with. There aren’t exact definition principle or rules of it. I think in the second one [Pause] more of a way of guiding students, whereas in the first lesson more of a way of kind of testing the students based on the information that’s just given.

Jack thought the questions guided the students in a way where they developed an understanding of **Fundamental Counting Principles**. When I asked him to explain how the questions helped students on their understanding, Jack explained and gave an example about their writing a question about fundamental counting principle in the second lesson.

In order for them to write their own question, they have to understand the concept beyond being able to have the certain specific situation and understand it. Understanding the situation and being able to work with problem and creating the problem, they have to understand the concept well enough behind it in order to formulate a question. It drives higher level of understanding for the concept. It is a good way of testing in and having them exercise the understanding of the concept for the lesson, I think.

I asked Jack whether or not he would like to revise anything in these lesson plans (bad and good). He was not sure what kind of changes he would make in the second lesson (good).

Jack: [Long Pause] I was given these lesson plans. I wanted to adopt it to my classroom. I probably would not make any major changes. I think the method used to teach, the concept for new spot on. I don’t know. I don’t think too many major changes, I don’t exactly know what changes I would make, if I did.

Jack observed that Mr. A gave the definition of probability and its formula in the first lesson (bad). The students were not given an opportunity to discuss what probability was. For this specifically Jack knew what he would change. He thought it would be better to begin the lesson with a discussion on what probability is. Then, he would utilize the chip activity. The students would learn **Theoretical and Experimental Probability** and compare these two.
It was better to start with a small discussion on what probability is. It jumps right into what the definition of probability was, what the differences between theoretical and experimental probability is. It’s kind of, went into an explanation, and then let’s use it for certain problems. So, if I were to revise that lesson, I would probably try to find some sort of activity to start the class with. That [activity] involved probability but don’t necessarily say this is probability, this is theoretical. This is experimental. [Pause] may be I have them I know they will roll chips in that lesson, and then they will be comparing their experimental to theoretical after they learn about experimental and theoretical, but I think possibly [Pause] may be having them do the test and come up with hypotheses to what they expect, and having them tested. And then from that kind of getting into the ideas of experimental to theoretical probability is and defining it possibly. After having the class as a group, from a discussion and define you know the concepts.

**Researcher’s observation about Jack’s analysis of the two lesson plans and their teaching**

Jack recognized the good and bad aspects of these lessons. However, he could not identify all of the important characteristics of these lessons. He knew that the first lesson was bad because Mr. A gave the definition of probability and its formula before the students investigated what theoretical and experimental probabilities were. The students were not given an opportunity to share and discuss their finding. Even though Jack watched the reflection of Mr. A, he did not notice that Mr. A did not prepare the lesson prior to his teaching. Mr. A decided how the lesson would go as he was teaching. Jack did not notice that during the activity Mr. A had the students collect the data, but did not let them discuss and solve the problem. The students were not given an opportunity to investigate all possible outcomes for experimental probability and find the relationship between the theoretical and experimental probability. Mr. A told the students the results of their experiment with the chip activity. He compared findings from 50 and 500 trials and gave the conclusion of it. Mr.A did not give students an opportunity to explore what they were doing, and to examine their findings and compare them with their classmates’ findings. Although Jack saw Mr. A give the definitions, he failed to see that Mr. A explained what needed to be learned from the activities such as theoretical probability, experimental probability and the relationship between the two concepts. Jack did not notice that the students did not discuss or share their findings.

Jack explained that the second lesson was better because the students investigated fundamental counting principles through the real life problems and were involved in solving all the problems. The students investigated *Fundamental Counting Principles* in different problems and saw the need for finding a short solution method such as tree diagram and multiplication of
all possible options. However, Jack did not notice the questions that Mr. A asked to motivate the
students as well as to have them check their reasoning and justify their answers. Another aspect
that Jack did not mention was that Mr. A had students collect the data, but throughout the
teaching of the second lesson (good), the students were asked to share and discuss their findings
with their classmates.

Preparation and Analysis of The Lesson Plan on Exponential Functions and Teaching
Episode

Summary of Jack’s lesson plan on exponential functions

Jack was given a lesson plan guideline that included objectives of the lesson (see
Appendix E). This lesson plan was prepared to teach a small group of senior high school
students at the Southern High School (see Appendix F).

The forty-minute lesson was prepared to teach to a group of high school seniors at the
Southern High School (see Appendix E). By the end of the lesson, the high school seniors would
be able to:

- Identify examples of exponential functions that were represented through a
  problem, equation, graph, or data table.
- Gain a general understanding of exponential functions’ rates and parameters.

Jack planned to use a chalkboard in this lesson plan. At the beginning of the lesson, the “light
versus water depth” activity would be given to the students for motivation. The students would
be assigned to work in groups of at least three and no more than four. They would be given a
worksheet to work on the activity. In groups, they would be asked to discuss the activity and
find possible answers. Then, the students’ thoughts and answers would be revisited later in the
lesson. After the light versus water depth activity, the students would be introduced to the idea
of exponential functions, specifically starting with equation of \( y = 2^x \). Jack would ask students
whether or not they think \( y = 2^x \) would grow quicker than \( y = x^2 \). He expected students to form
a hypothesis. Then, the students individually created a table with values ranging from at least -4
to 4 for both equations. After the students created the table, they would graph the equations and
formulate their conclusions. Jack wanted to ask the following questions,

- Do the two equations represent functions? Do the table(s)? Does the graph?
- Do you recognize an asymptote anywhere on either of the two graphs? Can you think of
  an equation for the line of the asymptote?
The students would draw the graph of $y = 4^x$ on the same Cartesian coordinate system where they drew a graph of the equation, $y = 2^x$. Then, the students would be asked to explain the graphical effect of changing the exponential base from 2 to 4. After this question, Jack planned to ask students to graph the equations, $y = 3^x$, $y = 3^{x^2}$, and $y = 3^{x-2}$ on the same Cartesian coordinate system. When the students graphed the equations, they would be asked to explain the graphical effect of changing the exponent from $x$, to $x + 2$ and $x - 2$. They would generalize the effect of “k” in an equation $y = a^{x+k}$. After examining the effect of “k” in equations, they would be asked to graph the equations, $y = 2^x$, $y = 2^x + 2$, and $y = 2^x - 2$ on the same Cartesian coordinate system. They would be asked to explain the graphical effect of changing the exponent from $x$, to $x + 2$ and $x - 2$. Then, the students would generalize the effect of “h” in an equation $y = a^x + h$.

After graphing these three equations, the students would be asked to graph the equation, $y = 2^{x+1} + 3$ and then one of the students would present his or her findings to the whole class. After this presentation, the students in groups would be asked to reassess their thoughts and give their answers of the light and water depth activity. The following question would be asked of the students, “Could the relationship between the amount of light and level of water be related exponentially? Why or why not?” By the end of the lesson, the students would make a final conclusion about the light versus water depth activity based on their knowledge of exponential functions. Jack also thought about the extension activity for this lesson. If time permitted, he wanted to include an interest rate problem. For assessment, Jack wanted to call on several students to explain their thoughts. The students would turn in their written responses of the light and water depth activity. They would also be given a homework assignment that would be due next class.

**Researcher’s observation about Jack’s lesson plan**

In this section, I wanted to provide my comments on Jack’s lesson plan on exponential functions. He was thinking of using the activity called, light and water depth to introduce the idea of exponential growth. Then, he planned to ask the students to draw the graph of $y = 2^x$ by plotting x and y values ranging form -4 to 4. He would ask students to compare the exponential function ($y = 2^x$) and square function ($y = x^2$). Up to this point, Jack was not thinking of
discussing definition and focus on exponential functions. I believe the students should know the
definition and properties of an exponential function before comparing different functions. After
the comparison of the functions $y = 2^x$ and $y = x^2$, Jack was thinking of having the students
draw the graph of $y = 4^x$, then asking them to draw the equations, $y = 3^x$, $y = 3^{x^2}$, and
$y = 3^{x-2}$ followed by the examples, $y = 2^x$, $y = 2^x + 2$, $y = 2^x - 2$ and $y = 2^{x+1} + 3$. As far as I
understand, Jack was planning to introduce the idea of exponential functions by an activity, and
then comparing the graph of square function ($y = x^2$) and exponential function ($y = 2^x$). The
next step in his lesson plan was to introduce horizontal (the $y = 3^x$, $y = 3^{x^2}$, and $y = 3^{x-2}$) and
vertical shifts ($y = 2^x$, $y = 2^x + 2$, $y = 2^x - 2$) and combination of horizontal and vertical shift
($y = 2^{x+1} + 3$). Up to this point, I did not see any clear discussion of the definition of the
exponential function or examples of exponential functions. For example, the exponential
function is for $y = a^x$, where $a > 0$, $a \neq 0$. Jack was not thinking of discussing these definitions
or giving examples for these definitions in his lesson plan. In addition, I believe it would be
helpful to divide exponential functions into two groups: exponential functions whose base is
greater then 1 ($y = a^x$, $a > 1$), exponential functions whose base is between 0 and 1 ($y = a^x$,
$0 < a < 1$). When the definition is discussed, the students should know or the instructor makes
sure they know that base cannot be negative numbers, zero or 1. Even though Jack did not
include the definition of exponential functions in his lesson, I was hoping that he would include
the definition of an exponential function as well as properties of exponential functions in his
teaching. Once the definitions and the two sub-types of exponential functions discussed, the
instructor should give the basic properties of an exponential function; for example, the
function, $y = 2^x$ where $x$ is restricted to rational numbers, if one coordinates several points for the
graph. The list of the points would be the following sets. The set, (-2, -1, 0, 1, 2, 3) is a domain
of the function consisting of the values of $(-\infty, +\infty)$. The set $(\sqrt{4}, \sqrt{2}, 1, 2, 4, 8)$ is a range that
included all possible values $y$ for $x$ in the domain $(0, \infty)$.  

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If one plots the points and sketches the graph, \( y = 2^x \) is an increasing function and its graph rises. When one moves from left to the right, \( y = 2^x \) will approach to the x-axis. Thus, x-axis is a horizontal asymptote. As \( x \) increases through positive values, the graph rises rapidly. The equation \( y = 2^x \) does not have x-intercept and the y-intercept is \( y = 1 \). The graph of \( y = 2^x \) is increasing on \((-\infty, +\infty)\). It is an increasing function. When it comes to explaining whether or not \( y = 2^x \) is a function, the horizontal line on the Venn diagram would show this a function. The equation, \( y = 2^x \) is one-on-one can be shown using the horizontal line test. The graph of \( y = 2^x \) is not symmetric. Thus, it is neither an even nor an odd function because a function \( f \) is even if the graph of \( f \) is symmetric with respect to the y-axis. Algebraically, \( f \) is even if and only if \( f(-x) = f(x) \) for all \( x \) in the domain of \( f \). A function \( f \) is odd if the graph of \( f \) is symmetric with respect to the origin. Algebraically, \( f \) is odd if and only if \( f(-x) = -f(x) \) for all \( x \) in the domain of \( f \). As \( x \) approaches \( \infty \), \( y \) values will get bigger as \( x \) approaches \(-\infty \). “\( y \)” values (the graph of \( y = 2^x \)) will approach the x-axis but it will never touch or cross the x-axis.

Then, the graph of exponential function whose base between is 0 and 1 should be given and properties of the graph should be discussed as a class. For example, the equation, \( y = \left(\frac{1}{2}\right)^x \) where \( x \) is restricted to rational numbers, if one coordinates several points for the graph the list of the points would be following sets. The set, \((-2, -1, 0, 1, 2, 3)\) is domain of the function consisting of the values of \((-\infty, +\infty)\). The set \((4, 2, 1, \frac{1}{2}, \frac{1}{4})\) is range that includes all possible values \( y \) for \( x \) in domain \((0, \infty)\).
Figure 4.38: Venn Diagram for $y = \left(\frac{1}{2}\right)^x$

If one plots the points and sketches the graph, $y = \left(\frac{1}{2}\right)^x$ is decreasing function and its graph rises. When one moves to the right, $y = \left(\frac{1}{2}\right)^x$ will approach to the x-axis. When one approaches to the left $y = \left(\frac{1}{2}\right)^x$ goes up. In terms of explaining whether or not $y = \left(\frac{1}{2}\right)^x$ is a function, the horizontal line on the Venn diagram would show this a function. The equation, $y = \left(\frac{1}{2}\right)^x$ is one-on-one can be shown using the horizontal line test. The graph of $y = \left(\frac{1}{2}\right)^x$ is not symmetric. Thus, it is neither an even nor an odd function because a function f is even if the graph of f is symmetric with respect to the y-axis. Algebraically, “f” is even if and only if $f(-x) = f(x)$ for all $x$ in the domain of f. A function “f” is odd if the graph of f is symmetric with respect to the origin. Algebraically, f is odd if and only if $f(-x) = -f(x)$ for all $x$ in the domain of f. As $x$ approaches infinity ($\infty$), $y$ values will get bigger as $x$ approaches negative infinity ($-\infty$). “$y$” values (the graph of $y = \left(\frac{1}{2}\right)^x$) will approach the x-axis but it will never touch or cross the x-axis. After this point, the students are ready to shift the function horizontally, vertically. I was hoping that Jack would mention the vertical shift between the two equations. In his lesson plan, he planned to use the following examples ($y = 3^x$, $y = 3^{x+2}$, and $y = 3^{x-2}$) for horizontal shift, ($y = 2^x$, $y = 2^x + 2$, $y = 2^x - 2$) and vertical shift, and a combination of horizontal and vertical shift ($y = 2^{x+1} + 3$). He planned to ask students to graph the equations by plotting $x$ and $y$ values. Although he mentioned that he would have a discussion to explain the graphical effect on horizontal and vertical shift, I was hoping he would compare these graphs and show students how the horizontal and vertical shift occurs.

Analyses of Jack’s lesson plan on exponential function
I analyzed Jack’s lesson plan and then I conducted an interview with Jack to talk about the planning process of his lesson. I asked him what was his first idea for this lesson plan. At first, he thought of the objectives and standards and he described the goals of the lesson. His goal was to have students find the graph of exponential functions and identify its behavior. His first idea was to have students investigate exponential function through a real life example activity called, light and water depth.

Well, going off the objectives, and standards that had to be addressed in lesson. I knew that it involved students working on exponential functions and they would by the end of the lesson be able to graph and identify data that display exponential behavior, and I knew that they have to be able to understand giving different parameters for an exponential function. Know how it would change the data. Change how the data is represented in graphs, that sort of thing. They would kind of have to be able to understand it in a context in some sort of [Pause] just you know this is an equation, this is graph. They have to understand it in a situation sort of real world application for the exponential equation. So, my first instinct was in order to motivate the students to get them into the lesson was to have an opening sort of problem where in groups they will be able to discuss it, maybe come to some sort of consensus and then kind of start the lesson there. So, I was looking for possible problems and then I found the one about the light versus the water depth. I thought that would be a good starting point.

After looking at the examples in his lesson plan, I asked Jack why he chose these examples for this lesson. He thought it was important for students to learn how different parameters affect the equation and graph. That was why Jack selected these examples following:

<table>
<thead>
<tr>
<th>Example</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = 2^x$</td>
<td>$y = x^2$</td>
</tr>
<tr>
<td>$y = 4^x$</td>
<td>$y = 3^x$,</td>
</tr>
<tr>
<td></td>
<td>$y = 3^{x-2}$, $y = 2^x + 2$, $y = 2^x - 2$ and $y = 2^{x+1} + 3$.</td>
</tr>
</tbody>
</table>

He planned to ask students to find the table values and graph them because he wanted students to learn basic parameters and how they affect equation and graph.

Jack: The reason I picked different examples was mainly because I knew that they have to know basics on different parameters and how they will affect the equation, the results, the data of the equation, how it will affect the graph. So, what I did was I started them out with $y = 2^x$ and, have them graph that. I wanted to make sure that they understood that $y = 2^x$ isn’t the same as $y = 4^x$ graphically. So, I wanted to make sure that they took each of these, equations made a table and then they graphed it. They didn’t use just their graphing calculator. Plugged them in that would be really easy. The reason I picked these specific examples was because each section I was having them change a different parameter. By the end, I gave them certain examples that involves couple of different parameters that were changed, and then have them [Pause] graph that problem just based on what they learn about changing parameter, how that affect the graph. I would have wanted them for this one instead of just putting into table values actually to take the graph and alter it based on what the parameter is given. So, they can
make a generalization about this parameter is changed. Graph is this much different. I mean it changes the data so the graph would look different. So, that’s why I wanted them by the end of the class; what they would do after that I wanted them to reassess their ideas about the opening situation of problem of light versus water depth and see if information that they kind of be able to go over during the period affected their ideas about the situation given.

I asked Jack to explain his goals for asking these questions. He thought the group of high school seniors would make generalization about how the parameters affect the graph of the functions.

I mean what I would have them do is each of the examples. I want them to have a table value, have them plug in values, and then graph the values. Take those points and put into equation or I mean coordinate system. [Pause] I was hoping with each of these different sections they will be able to make a generalization that if you change with one parameter, and then this is what happens graphically to function. I hope from the generalizations by just looking at the equation, they know how it would affect the data graphically.

As can be seen from the excerpt above, Jack only put equations for the examples. He expected students to create table values for x and y and graph it. I asked Jack he found these examples and activity for this lesson. He explained that he searched on the Internet to find these examples and the light and water depth activity.

I pretty much went to Google and typed in lesson plans on exponential functions and I came up with this NCTM lesson plan that actually was fairly in depth and covered several different class periods or they actually went out and collected data certain classes they analyzed. So, I kind of just modified back the first lesson in by taking the example that they used at the first lesson. That’s what I used for.

I wanted to learn more about how he selected these examples. So, I asked Jack why he chose this activity rather than some other examples. He thought that light and water depth activity would be better then an interest rate problem. According to him, this activity was interesting and it would help him to create a group discussion.

So, hopefully it would motivate students to be more interested in the concepts. So, kind of why I chose this specific one just because it wasn’t just a [Pause] I could have picked one on interest rate possibly. I guess you can give a certain situation and context I thought this would be more interesting, more students would [Pause] I don’t know be able to just kind of discuss the information no more than saying okay you’re going out to buy a car, house. This is the interest rate given by the bank and what the interest amount of time would be. I figure this might be an easier, not necessarily easier, more gear towards a group discussion. Having them discuss the situation. Might be able to better that. That’s why I picked this specific example.
Jack thought the high school seniors would notice the exponential behavior in the interest rate problem. He preferred to use the light and water depth activity because he thought that the high school seniors would not see that this activity would be related to exponential behavior. If they started out they may not necessarily expect to look like an exponential graph because they may not encounter an exponential graph until this lesson. So, probably by the end of the lesson, they’ll be like well it’s math related. It’s numbers. It’s probably just an exponential graph. But with the situation [light and water depth], I figure it doesn’t lean itself specifically towards exponential sort of relationship between light and depth. So, I figured it initially creates more controversy but wouldn’t be as directly linked to an exponential sort of concept. That’s the reason why I picked it.

Jack was asked to explain how he set his goals for this lesson. From the given objectives, he knew what he was expected to do for this lesson. So, he did not have difficulty writing his goals. Well, from the objectives I knew basically what I had to cover or what the students should be able to do by the end of the lesson when I was writing the lesson plan or thinking of ideas for what the students should be doing during the lesson. I just kind of followed well, they should be able to do this by the end of the lesson that I have to make sure they are actually doing it in the lesson, working with the concept and doing it.

**Researcher’s observation about analysis of Jack’s lesson plan**

Jack’s weak subject matter knowledge of exponential functions and limited understanding of relationships among different representations prevented him from making appropriate instructional decisions to meet the students’ needs. In addition, the necessary steps (definition of an exponential function, properties of an exponential function) to introduce an exponential function are missing. Jack was told that the students did not learn these important concepts in their previous classes. He should have included more examples about exponential functions, such as where “a” is between 0 and 1 (0.0001<sup>+</sup>, 0.9<sup>+</sup>,) and “a” is greater than 1 (1.0001<sup>+</sup>, 2.006<sup>+</sup>). However, “a” cannot be 0 or 1 (a ≠ 0 or a ≠ 1). If I were Jack, I would give examples to show what values “a” can or cannot be. I also believe that it’s very valuable and helpful for students to discuss basic properties of exponential function as well as two basic types of graphs (a<sup>+</sup>, a > 1 and a<sup>+</sup>, 0 < a < 1). In addition, Jack should have thought about how he would introduce the light and water depth activity to the students and how this activity would make the students involved in the process of finding the relationship between light and water depth.

**Summary of Jack’s teaching**
Jack taught his forty-minute lesson to four female and two male high school seniors. He started out with writing the question of light and water depth to the board. After he finished writing, he introduced himself to the students, and then read what he wrote on the board “Have you ever noticed how the amount of light differs the further you are under water? Consider how the light intensity changes from the surface of the water to the bottom of the ocean.” Jack just told the students to get into groups of 2 or 3 and find the graph of light versus water depth. When they were working on the activity, he walked around and checked their work. He approached one of the students and asked why he thought it was a curved line rather than a straight line. Then Jack asked the student if he was familiar with the term “asymptote.” The student did not respond to the question, so Jack asked the same question of whole class. He said, “the graph is trying to reach, but it would never reach. You cannot have total absence of light.” After this explanation, he let the students work on the activity a few minutes in their groups, then moved onto a new question.

Jack approached the students with the following question, “have you seen an equation look like this $y = 2^x$? Today, we will learn about those types of equations, and learn about graphing these.” Jack asked the students how they would graph this equation. Then he created a table values for $y = 2^x$. Jack wrote all the x values (-2, -1, 0, 1, 2), and asked the students to find the y value using x values. Again, Jack asked the students to find the value of $y = 2^0$ when $x = 0$. Some of the students thought it was 1, others thought it was 0. Jack told the students it was 1. After answering the question about zero exponents, he continued writing and filling in the table values for y. Then, Jack drew the graph of $y = 2^x$ using table values of x and y and told the students the graph was a representation of the equation ($y = 2^x$). By looking at the graph, the students were asked to determine whether or not the graph was a function. The following excerpt shows how Jack and one of the student discussed the answer of this question,

Jack: This is the graph of exponential equation. Did you guys know whether or not this would be a function? Just looking at the graph.
Student: Yes.
Jack: Why do you say that?
Student: Because it passes the vertical line test.
Jack: Yes, you can use the vertical line test. Does it pass the vertical line test? Vertical line test, if you take any vertical line, just looking at the graph. Any vertical line, draw it through the line. It should not pass more then once [pointing the x-axis]. [He drew a parabola opening to right], and you drew a vertical line right there. It would hit twice, it would not be a function.
After talking about the vertical line test, Jack asked students to compare whether or not the graph of $y = x^2$ would rise faster then the graph of $y = 2^x$ in graphical form. He created two tables for $x$ and $y$ values of each of these equations and drew the graphs. As Jack was explaining how these two graphs were similar, he decided to include the value of $x = 3$ in tables to determine if the graph of $y = x^2$ rises more then the graph of $y = 2^x$. Jack asked students to graph the equation, $y = 4^x$ in groups of two or individually. As they were working on the question, he took his calculator to graph $y = 4^x$. At the board, he drew the graph of $y = 4^x$ on the same Cartesian coordinate where he drew the graph of the equation, $y = 2^x$. He wanted students to compare graph of $y = 2^x$ with the graph of $y = 4^x$. During this time, he checked a couple of students’ work and talked to them about their solutions. He asked one of students to draw the graph of $y = 4^x$. Using the table values from her notes, the student drew the graph of $y = 4^x$. Jack asked the students if they understood the difference between these two graphs ($y = 2^x$ and $y = 4^x$). They did not answer. Jack moved onto the next example and asked students to graph $y = 4^x + 2$. They were given a couple of minutes to work on graphing the equation. Then, Jack asked one of the male students to show what he found at the board. First, the student created a table for $x$ and $y$ values. He drew Cartesian coordinate and plotted the points. Then he drew the graph of $y = 4^x + 2$. Jack asked the whole class where the asymptote would be on the graph. After asking the question, Jack answered and showed the students how asymptote get close to the x-axis. The asymptote line would be at $y = 2$

Jack wrote another example, $y = 4^x − 2$ for finding asymptote. He asked one of the female students to find the asymptote line of the graph of $y = 4^x − 2$. The student thought the asymptote line was at $y = −2$. Jack asked the whole class if anyone disagrees with student’s answer. The students were quiet and did not give any response. For the last few minutes of the class, Jack went back to the light and water depth activity that he gave at the beginning of the class. He asked the students whether or not they wanted change or to keep their answer. Then, Jack asked one of the female students to show her answer at the board. The student correctly drew the graph of light versus water depth. After the student finished writing her answer, Jack explained the behavior of the graph and the relationship between light and water depth. Then, he asked the students if they had a different answer. They were quiet and did not say anything.
After the light versus water depth activity, Jack asked one last question. He drew the graph of $y = 2^x$ without plotting the points for x and y values and then he asked students to draw the graph of $y = 2^x - 3$. He gave students about a couple of minutes to work on graphing the equation. Then, he drew the graph on the board and asked the students if it was a function. One of the students said, “It was, because vertical line test shows.” Then Jack ended the lesson.

Analyzes of Jack’s teaching

After his teaching of the lesson on exponential function, I interviewed Jack to ask him to evaluate and reflect on his teaching. Jack found that it was difficult to motivate the students. Earlier, he thought the activity (i.e., light and water depth) and the examples would be very motivating. During his teaching, he noticed that the activity and examples were not interesting to the students. In addition, it was not easy for him to interact with the students. Teaching the lesson did not go the way he had expected, but he still continued to teach the lesson as he had planned earlier.

Jack: I think it was a moderate success some ways. On a base level, I feel like kind of connect to the students as much as I had hoped. I didn’t really know their names, even with six of them. [Long Pause] I think on the base level I felt like kind of hard to connect to the students. Try to motivate them with the material. I don’t know because number one I didn’t know their names. They could introduce themselves to me, but I wouldn’t remember their names more then one or two of them. It kind of takes me a while to learn. So, I mean as far as that goes, I mean there is more to motivating students then to be able to call their name that sort of thing. On a base level, that’s usually a good way to interact with the students. I know I wasn’t able to go over everything that I planned to go over. All the different parameters and everything that I’ve planned ended up not necessarily working out everything. I tried to go over as far as I could and when I realized that we are getting towards the end of the class, I tried to instead of like leaving them with possibly still questions. Never mind I try to close it bringing the example back like I planned to close it. So, I just took out some examples [Long Pause] I figured if Mr. A leading to go over the lesson again whenever I hope he gave like a good introduction for the students themselves.

I wanted Jack to reflect on his teaching in detail. So I showed him a video clip from his teaching, showing when Jack asked the students to graph $y = 4^x + 2$ and then asked them to find the asymptote and its line. Only one out of six students were able to identify what an asymptote was. Jack knew the majority of the students did not understand it, but he still showed where the asymptote line on the graph, and moved onto another question. However, Jack thought that one or two of the students understood what an asymptote was after he explained it to them.

Interviewer: What did the students say when you asked them?
Jack: Basically I got a lot of blank stares from all the students. I’m not even sure they knew what an asymptote was like, if they ever studied before exactly. I think being at the class, I kind of asked them. I think there are one or two of them understood what it was after I explained it. I don’t think any of them recognized the word and were able to put it to a concept. Once I explained it a little, there were a couple of them they are like yeah I understand what it means. Maybe learned in a previous class. I don’t know something. At this point, in the lesson, I kind of to remember them all having blank stare when I asked them, if the graph is this, what do you think happens to asymptotes? I did not get too many responses. So I tried kind of guide them into if the graph goes up to 2 unit, what do you think that asymptote does. I think the one girl kind of basically just said it probably goes two units. I think that’s what she said.

I asked Jack to explain how the students had difficulty with exponents. I wanted to learn how Jack dealt with the students’ difficulties. I observed that they were giving different answers to the value of negative exponents. When Jack was creating a table values for x and y, he realized the students did not have a good understanding of negative exponents. He just showed them how he found it at the board \( 2^{-2} = \frac{1}{2^2} \). He assumed the students helped each other finding the value of negative exponents. He did not want to explain the negative exponents in details because he wanted to continue teaching the lesson.

Jack: Most of the students it seems they were able to graph it. I think most of them were able to kind of go along with the lesson, plug in points, if they didn’t exactly understand, what does that mean. How is \( \frac{1}{2^2} \). So, they are able to help each other as far as that go.

Interviewer: Do they have any difficulty working with exponents?
Jack: As far as having negative exponents that’s where they run into problem. That took little bit of working through explaining that. And in the same respect, I didn’t exactly want to go into like an algorithmic sort of explanations. I did not want to go straight into what I knew in order to teach the lesson. I basically have to give this to the students that in order to progress the lesson, which I mean I want to be the ideal situation ever, but I felt that was needed at that time.

As can be seen from the excerpt below, Jack showed the students how to find the value of negative exponents, but he did not check their understanding of them. Jack gave the rule for finding the value of negative exponents and moved on to the next example.

In the video clip, Jack used graphical representation to find the asymptote. He thought that using a graph would help students develop their understanding.

Interviewer: How did you find the asymptote line for this question \( y = 4^x + 2 \)?
Jack: I changed to \( y = 4^x - 2 \). I was asking them right there. What they thought would happen if you change it to subtract thing in the equation. Basically I was trying to get
them to visually find the asymptote, kind of look at the lines and look at the directions of the line, what the lines represent graphically. Just kind of visually find it. There is a way of algebraically determining asymptote but I figured since some of them knew and recognized what the asymptotes were instead of going through a lot of algebraic details. Just kind of having them visually look at it. It would be more for their understanding of the concepts.

Jack failed to recognize that the students did not have a good understanding of what an asymptote was. He thought it would be more appropriate to use graphical representation to introduce the asymptote line. However, he did not notice that only one student was able to draw the asymptote line. Toward the end of the lesson, he decided to go back to light versus water depth activity. As he had planned earlier, Jack wanted to discuss the students’ findings. According to him, they had correct answer by the end of the lesson. So, he could not have students discuss their findings. During the lesson, he asked one of the students to draw the graph of light versus water depth. Rather then asking the student to explain it, Jack explained the relationship between the light and water depth. Then, he asked students if they had a different answer. Since the correct answer was at the board, the students did not say or explain their reasoning. During the interview, Jack explained that most of the students had a correct answer for the light and water depth activity. He was unable to have them explain their reasoning.

Interviewer: You gave the question at the beginning of the class, and decided to discuss it at the end of the class. Why did you do that?

Jack: I basically formulated where I thought it would be more interesting for the students to kind of see possibly their views change just given a lesson instead of having it at the beginning of the lesson. I figured, if you give it at the beginning where they possibly don’t have any understanding of the exponential functions, they set a hypothesis in that sort of thing. Throughout the lesson, they possibly come to reassess their initial thoughts by the end of the lesson. Maybe change the original conclusions and I thought it would be more interesting for the students. Because it’s not only just kind of either expect them to know what is coming in a lesson or expect them to regurgitate what’s going on in the lesson. I think it’s more engaging for the students that way, setting up that way. My problem was that most of the students were actually like trying to be involved and ended up with right answer any way at the beginning of the lesson. So, by the end of the lesson, yeah you guys still think. I tried to get them to kind of explain it. It won’t work out. I don’t know.

In this activity, Jack noticed one of the male students had a different answer. When he asked if the students had different answers, this student was quiet. Jack did not want to call on him to show his answer. This student was quiet and did not say Jack explained the relationship between the light and water depth and then asked the students if they had different answers.
Interviewer: Did any of the students have a different answer?
Jack: There is the one guy ended up I saw the one who wasn’t participating at all. He was kind of sitting there. I saw him draw a straight line at the beginning of the class. I knew he had a different answer. By the end of the class, when I was asking them did you have any that didn’t look this. I didn’t specifically know his name and he hadn’t been participating at all in the lesson. I felt like it would be out of place to ask him to show it [his answer] at the board. Especially when would be in front of the kids who had the same thing at certain time. Number one, I knew I was running out of time. Number two, I don’t know. I thought it could have opened up for a nice discussion. I didn’t want to call him out on it. Especially, when I asked does any one have a different something from this. He didn’t respond at all. I didn’t put him through that. I mean it should be done in some cases. Obviously showing students to come up with different work and I think that’s a great way of learning. Like having different answers, having students to discuss why one probably is better then the other or which one is more right. What does it mean? I think it would be more advantageous for the students’ understanding but at that particular time point, I just had a problem with calling on him.

I asked Jack whether or not he would change anything on this lesson if he was asked to teach it again. He wanted to include technology to teach exponential functions. He wanted students to learn graphing, creating table values in exponential functions. After learning about exponential functions, the students should use graphing calculator. However, Jack did not indicate how he would use technology to help students better understand exponential functions.

Jack: If I had go back and change something in my lesson plan, it would be possibly including some sort of technology in it to where the students would work with a couple of problems where they actually plug in the points. They come up with their own graphs, and they kind of understand it, but beyond that when you start moving to different parameters instead of them graphing, showing it to each of them. I guess questioning, asking them to come up with if we change the parameter, what do you think the graph is going to look like? Showing them [students] how the data looks like, all that sort of thing. It’s a bit more efficient then just having the students continually put values into equation to graph it. Like having work them couple so that they’ve been doing that. Possibly even have them use their own graphing calculator, get them to groups and giving them a list of equations, ask them how it changes. Based on that parameter, what they think, instead of having where the teacher up there standing and showing them in projector with a graphing calculator. Having a worksheet or something where they have a list of equations, they kind of look at it. This would be after they work a couple of problems by themselves, by hand.

When I asked Jack to evaluate his teaching and determine whether or not he achieved the goals for this lesson, he thought he had accomplished his goals. However, Jack was not sure how well
the students understood the relationship between the different parameters and behavior of the graphs.

I think the goals [Long Pause] [He read the goals from his lesson plan] I think the word ‘general’ is kind of weak, but I think they have a better understanding than they did going into the lesson, most of them. So, in that aspect, I met that goal. They were able to graph exponential functions in most part. We [Jack and the students] didn’t get so much into being able to identify data that displays exponential behavior. I mean they were able to [Long Pause]. I had at the beginning and end of the class, but they were able. I’m not sure necessarily they were able to relate to exponential data, but they understood what the line look like. Hopefully, from that, they were able to understand the example of data that represent exponential function.

In his teaching, Jack asked students whether or not the graph of \( y = 2^x \) was a function. For his explanation, he used the vertical line test. Jack was asked why he used the vertical line test on this question. He was surprised that the most students did not recognize the vertical line test. The following excerpt explains his reasoning:

Because in one of the examples I asked them would this represent a function, and one of the kids answered yes. I asked him why it’s a function. Well, look at the vertical line test. Well, I went to explanations of it a little bit because I got a few blank stares or there were a couple of people like I don’t understand that. It kind of caught me off guard. Usually the one thing people know is the vertical line test.

Jack thought letting students work and discuss their ideas in groups or individually was very helpful for students. However, it would have been better if he had a whole class discussion about students’ findings. Most of the time when Jack asked questions, they were quiet. He ended up answering his questions.

Interviewer: How was the communication between the students?
Jack: I felt that when they were graphing or doing different equations by graphing, plugging points and coming up with data tables, whenever they needed help, they were able to talk to each other, get the help they needed. If they were actively involved, there was the one kid who is basically sitting there, writing a little bit on the paper for nothing. He was basically off on his own. Other people were definitely like talking and helping. I tried to fit in where I thought it would help the students, but the most part I tried to make sure it was right they were helping each other.

The majority of the students were able graph the relationship between light and water depth. However, Jack was not sure whether or not the students had a good understanding of how they found the relationship between light and water depth.

Interviewer: The majority [of the students] was able to draw the graph. Were they able to explain how they found it [Light versus water depth]?
Jack: I think a couple of them came up with explanations nothing too detailed. I don’t know. I tried to get them to expand, talk little more about it what it means. I’m not sure they understood how they came up with the graph.

Interviewer: When the students found the graph of it, were they able to say what kind of function it is?

Jack: No, at the very beginning, they were probably no. I don’t think they were. I never asked the question what they thought what the graph looked like or what kind of graph it looks like, but I think they are probably able to call it exponential after we had it in the lesson. I never specifically asked them this question. That’s a good question definitely. It was a good experience even if didn’t necessarily get every goal done.

**Researcher’s observation about analysis of teaching of the lesson**

Jack did not even talk about how to draw $y = 4^x + 2$ using graph of $y = 4^x$. Without discussing the behavior of the function, he asked the students what the horizontal asymptote was. He should have discussed with students how the function behaved as $x$ approached negative infinity and positive infinity (to the right, to the left). For example, it could have been helpful to ask students whether the function touched the $x$-axis when you went from right to the left. This clearly shows that his limited subject matter knowledge made it more difficult for him to ask appropriate initial questions or explanations that made the concept easier for the students. The impact of his limited or weak subject matter knowledge and pedagogical content knowledge on his instruction was very obvious in his teaching. When he began the class he said they were going to learn the graph $y = 2^x$ instead of what this function was called and discussing how these functions are defined. Jack’s idea of making a table and substituting values for $x$ to draw the graph of $y = 2^x$ was very appropriate. However, when he drew the graph, he did not mention any properties of this function (increasing, decreasing, domain, range, horizontal asymptote, symmetry). I was hoping that at this point he would go back and discuss the definition of exponential functions, but he preferred to compare the exponential ($y = 2^x$) and square functions ($y = x^2$) and as well as the difference between those two graphs. Then he asked students to draw $y = 4^x$ on the same coordinate plane and to compare the two graphs ($y = 2^x$, $y = 4^x$). I believe this was a good opportunity for him to tell the students that the graphs of exponential functions would look similar as long as the basis is greater than 1. However, he did not mention this property. This indicates the lack of pedagogical content knowledge. He asked the students what the difference between the graphs was, but he moved on to the next example without discussing...
anything. Up to this point, his teaching was teacher-centered because he could not ask appropriate questions to get students involved or to discuss the exponential functions.

The next example was the graph of $y = 4^x + 2$. Jack erased the board and drew the graph of $y = 4^x + 2$ without any reference to $y = 4^x$. In other words, he should have shifted $y = 4^x$ two units up to draw the graph of $y = 4^x + 2$. At this point, the students did not know the definition of exponential function properties of exponential function and the graph of exponential functions whose base between is 0 and 1 ($0 < a < 1$). Jack wanted to draw the graph of $y = 4^x + 2$. He substituted values for $x$ and plotted those points to draw the graph. Again, he did not mention the vertical shift or the relationship between $y = 4^x$, $y = 4^x + 2$ and $y = 4^x - 2$. At this point, I was hoping that he would give an example of exponential function whose base is between 0 and 1. However, he preferred to discuss the activity. He asked one of the students to draw the graph representing the relationship between light and water depth. After going over what the graph looks like, he drew the graph of $y = 2^x$ and asked students to draw the graph of $y = 2^x - 3$. He did not even discuss the difference between these functions (y-intercept).

During our interview after his teaching, he watched the video clip and told me that his activity was not interesting enough for the students. He could not communicate with students or get them engaged in the class. He said even though the students did not understand what horizontal asymptote was he continued the lesson because he thought it was necessary to finish the lesson as planned. He mentioned the students’ difficulties when he was calculating negative exponents of numbers ($2^{-1}, 2^{-2}$).

Jack believed that he reached the goals of his lesson plan. When I asked him what he would change if he taught this class again, he said he would use more technology. However, during evaluation and reflection of his teaching, he did not talk about the organization of his lesson plan (introduction, examples, activities, etc.). Throughout the class, he drew the graphs by plotting points (point-wise approach) and did not mention the relationships between his examples ($y = 2^x$, $y = 4^x$, $y = 4^x + 2$, $y = 2^x - 3$). He did not even define the behavior of the exponential function, even though he noticed that the students did not understand the asymptote. He did not know why they did not understand it. Jack should have asked the question, “How does the function behave as $x$ approaches positive infinity and negative infinity?” His approach to the graph $y = 2^x$ was appropriate but he did not discuss the behavior of the function for very
large or small x values. I don’t think students knew what asymptote was. If the students knew that the function approaches the x-axis, but did not cross or touch any x values. It would have been easy for Jack to explain the concept and engage the students in the class.

Jack had taught 3 times in his previous method class. Basically, this was his fourth teaching experience. I believe that one of the biggest factors that prevented him from communicating with students or getting students engaged in the class was his inexperience. Jack is a novice teacher, and he will learn and improve his teaching skills by teaching.

**THE CASE OF SARA**

Sara is a 21-year-old university senior who has been attending the Southern State University and majoring in mathematics education. The secondary mathematics and middle grades mathematics teacher certification programs are approved by the State Department of Education and are credited by the National Council for Accreditation of Teacher Education. A student preparing to teach secondary school mathematics have to take at least twenty-one (21) semester hours of mathematics, statistics, and/or computers beyond the common degree prerequisites. She has a 3.8 overall college G.P.A. She stated, “Mathematics has always been a subject that I have enjoyed.” Her earliest memory of mathematics involved struggling with her homework in kindergarten. Sara found that “learning mathematics is always the same. As long as you do your homework and make an effort to figure things out on your own you will learn.” For her, calculus was no more difficult to understand than division.

Sara changed her major from civil engineering to mathematics education last year. When she decided to change her major, she questioned herself about why she wanted to teach mathematics. At first she was not sure, even though she enjoyed her mathematics course. She truly believed that every student should learn at least a basic level of algebra. The fundamental reason for teaching mathematics to students is to help them function in society. Mathematics is very important in every aspect of life such as calculating a budget and balancing a checkbook; it is necessary in different business, as in economics, statistics. In addition, the students who live in this society major in different fields such as science, medicine, and engineering. As a teacher candidate, she wants to make sure that students have a strong understanding of mathematics.

Sara is a bright, friendly and talkative person who is willing to be a mathematics teacher. When I ask her why she likes mathematics, she responded,
I really like it. Why [Pause] it comes easily for me, but I have to work hard to get it. I am able to understand concepts and like, exciting [Pause]. I really like calculus how [Pause]. I was taking calc base physics when I was taking calculus. So those went hand in hand. You were learning about the velocity and all these things. It was interesting to me how the physics, and all that works. Like taking modern algebra, elements of algebra. [Long Pause] I know it is about doing proofs and stuff. Going through trying to figure out this problem and you get stuck and, you know eventually you are able to figure it out. You just get all excited. So that’s why I enjoy it. I feel like we need to know the applications, relational understanding of what math is for. And so I just feel we need to take physics if we are going to take calculus.

Sara likes to learn mathematics and see how it relates to other disciplines such as physics and to real life. She took geometry, algebra II, trigonometry and analytical geometry in high school. In college, she completed several mathematics courses as well. These courses were pre-calculus, trigonometry, calculus I, calculus II, calculus III, linear algebra and modern algebra. At the time of the study Sara was enrolled in an elements of algebra course as well as in two method courses (teaching mathematics in middle grades, and how adolescents learns mathematics). When I asked her about the method courses that she was currently taking at the time, she was not sure how they would be helpful to her before she teaches any courses.

Sara was concerned that she would not remember what she had been learning in method courses when she starts teaching during her internship.

Sara: Well, I feel like they are good, but I am not going to be doing my internship for another year and a half. I know I am not going to remember any of this stuff. I am taking notes, but I just feel like [Pause] I don’t know it’s almost pointless to try to teach us this stuff right now. Long before we get into the classroom. Or I guess you know it could be helpful like, teach us now, we will keep it fresh eventually. May be down the road, I’ll remember this from this class. [Long Pause] Trying to teach us now that we can’t even apply it at all. [Pause] There is so much information. All this information for us to learn, you try to read it, but you can’t remember all that stuff. It’s so much information, that’s one problem I have with it. The stuff we learn is good. Examples we learn about how to teach. About different things like, little models represent mathematical ideas. I think those are good for us to understand and to know how to use until teach with.

Over the past year, she has been tutoring different grade level students (middle, high and college students) for mathematics. However, her tutoring experiences are mostly been with high school students. Sara also works as a grader for a calculus I class. Throughout her tutoring experiences, Sara has encountered students with many different levels of mathematics proficiency. Most commonly, she saw that the students struggle with learning algebra. That is
why she wants to teach algebra rather than calculus, which she had originally anticipated. She plans to teach high school, but she is getting both a middle and high school certificates.

The Function Questionnaire

Question 1

Sara was asked to give an example of a mathematical function. She wrote an algebraic equation \( f(x) = x + 1 \) as an example. Then for the example, she wrote, “A line is a function.”

Question 2

Sara was asked to give a non-example of a function. She wrote an algebraic equation \( x = 3 \) and drew a line vertical line that passes through the x-axis as an example of non-function (see Figure 4.39). Then for the example she said, “if the graph passes through a \( x \) coordinate more then once it is not a function.”

![Figure 4.39: Sara’s example of a non-function](image)

Sara wrote an equation for an example of non-function. She could not explain why the equation was not a function. She used the vertical line test to determine that the graph of the algebraic \( (x = 3) \) was not a function. The following excerpt illustrates her dependence on the vertical line test.

Interviewer: Why did you choose these examples rather than something else?
Sara: Well, this was just my reasoning. I just thought of any line because I know line is a function. The first thing that pops into my head for number one for any function and for non-function, I knew that you couldn’t have, well vertical line. If you do the vertical line test, you can’t have two different points with the same coordinates. So, I said \( x = 3 \) which is a vertical line that you know has an infinite number of points for same x coordinate, then definitely not a function in that case.
In questions 3-7, I asked Sara to identify whether or not the relations represent a mathematical function (see Figure 4.40)

<table>
<thead>
<tr>
<th></th>
<th>Identify whether or not each of the following represents a mathematical function.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.</td>
<td>$2x - 3y = 5$</td>
</tr>
<tr>
<td>4.</td>
<td>$x^2 - y = 5$</td>
</tr>
<tr>
<td>5.</td>
<td>$x^2 + y^2 = 4$</td>
</tr>
<tr>
<td>6.</td>
<td>${(1,2),(2,3),(3,4),(4,5)}$</td>
</tr>
<tr>
<td>7.</td>
<td>${(1,2),(-2,4),(3,4)(-1,1)}$</td>
</tr>
</tbody>
</table>

Figure 4.40: Identifying mathematical function

Questions 3

For question 3, Sara wrote the question in slope intercept form to determine whether or not it was a function. After writing the equation, $y = \frac{2}{3}x - \frac{5}{3}$, she knew that it was a non-vertical line. I realized that Sara had a limited understanding and use of vertical line test. Sara identified that graph (i.e., $2x - 3y = 5$) passes the vertical line test and represents a mathematical function.

Interviewer: you said, “This is a non-vertical line.” What does it mean?
Sara: Well, any line is a function unless it’s vertical. The only line that would not pass this vertical line test is the vertical line. Since it has a slope other than does not exist then it’s a function.
Interviewer: In this case, did you look at slope then?
Sara: [Long Pause] I think so, I guess without even thinking about it because I took it out of the standard form and put it into slope intercept form, so that I could be sure that it has a slope.

Question 4

Sara solved $x^2 - y = 5$ for $y$ and wrote $y = x^2 - 5$. She knew that it was a quadratic function. The graph of the equation would be a parabola. Then for the example, Sara wrote, “It never passes through the same x coordinate twice.” However, she did not try to solve the equation algebraically or numerically to show that x values cannot have more than one y values.

Interviewer: How did you determine this function?
Sara: I recognized what kind of function it was. It was a line with some sort of slope. So, I knew by the nature of the line that it would pass the vertical line test. So, I didn’t try to picture the graph and think about the points and passes the vertical line test. I just,
okay it’s a line, it’s a function, passes this test. With that number 4 [question 4], parabola I saw it quadratic. I saw that would be a parabola. So I knew that because it’s a parabola, it would pass the vertical line test.

**Question 5**

Sara immediately recognized that $x^2 + y^2 = 4$ is a circle and a circle would not pass the vertical line test. The following excerpt indicates her tendency to visualize the graphs of equations.

Interviewer: you said, “No, this is a circle. It does pass through the same x coordinates twice.” How did you know this is a circle?
Sara: Because I’ve been taught what the equation of a circle looks like. So, I recognized that is a circle $x^2 + y^2 = 4$.

**Questions 6-7**

Question 6 and 7 were given in set notations. My goal was to investigate how Sara would interpret different representations of functions and non-functions. On a coordinate plane, she plotted the points $(1,2)(2,3)(3,4)(4,5)$ by using a dot. Then, she connected the set of points with a continuous curve (see Figure 4.41). She wrote, “No, this is a set of points. It is not continuous and therefore, not a function.” She looked at whether or not the relationship between the values of x and y represent a function. Sara was not sure whether or not the set notation would represent a function. For Sara, the relationship between the values of a set of points could not be a function. If the set of points are grouped for values of x and y, then it would be called a function. She did not see that functions could be continuous and discontinuous.

![Figure 4.41:Sara’s graph for question 6](image-url)
Sara: I guess it probably. They [set notations] should probably be connected because. [Pause] I have never really thought about it before but I know that I have seen where like I just said they have different points. They ask if this is a function and you know if you have your x’s 1, 2, 3 and 4 and then the y’s that they 2, 3, 4, and 5. So, all the x’s you know just go to one y value so, by that you would say it’s a function, but I never really question it. These are a couple of points should be connected or what, but I assume that it should be connected. Probably it would be a function. Because it would make sense if they’re broken up and called it function.

In question 7, Sara was asked to find whether or not the relation represents a function. At first, she wrote, “No, this also a set of points not continuous.” Then, she plotted the points 
\{(1,2)(-2,4)(3,4)(-1,1)\} by using dots (see Figure 4.42). After drawing the graph, Sara referred to question 6 and talked about her answer. She decided that the set notation in question 6 was a function.

Sara: I just took reasoning from here, but it is a function. Yes, if the points are connected, I guess it is a function because you never repeat any of the x values, they are all you only have, one x value dealing with one y value.

Figure 4.42:Sara’s graph for question 7

When I asked Sara whether or not the relation in question 7 was a function. She was confused, but she accepted questions 6 and 7 as representation of a function.

Sara: Well, I see yes, it is a function for both of them. Even though I’m slightly confused because I don’t really, I mean. If it was just four points, then it couldn’t be a function. If it’s function containing those points, then it can be.

**Question 8-9-10**

In questions 8, 9 and 10 (see Figure 4.43), three graphs were given and Sara was asked to determine whether or not they were functions given in graphical representations. She immediately answered these questions. She said that question 8 and 10 were functions because
the graphs would pass the vertical line test, and that question 9 was not a function since it would not pass the vertical line test.

![Image](image_url)

**Figure 4.43: Questions 8-9-10**

By asking questions 3 though 10, I wanted to investigate Sara’s solution strategies as well as her interpretation of different representations of functions and non-functions. Questions 3, 4, and 5 are given as equations, questions 6 and 7 in set notation, and questions 8, 9 and 10 in graphical representation. I observed that when she was given relations in equation form, she tried to determine what their graphs looked like in order to use the vertical line test. She did not think of solving the relations algebraically or numerically.

For example, she could have substituted values for x or y in the equation \(x^2 + y^2 = 4\) to show that some x values could have two y values. She also used vertical line test for the questions 8, 9 and 10. However, she did not explain why the graphs were or were not functions nor did she mention the univalent property of functions (Even, 1990) in her definition.

In question 8, Sara was asked to determine whether or not the graph was a function. She said, “Yes, this passes the vertical line test and is continuous.”

Interviewer: What do you mean when you say this graph is continuous?
Sara: [Pause] I mean that it has never any. [Pause] Any places where the graph stops and starts up somewhere else, and it doesn’t have any jumps or any places where breaks or where it has to be like smooth. You know there can be sharp edges like, angles.
In question 9, Sara drew a vertical line on the graph and said, “No, this does not pass the vertical line test.” However, she failed to explain what it meant when a vertical line would intersect the graph at two points. When I asked Sara how she would respond if a student said, question 5 and question 9 were functions, she assumed the student did not understand the vertical line test and he or she was confused with \( y^2 \). However, Sara did not think of using a different approach or try to explain why they were not functions.

Interviewer: How would you respond or explain to someone who says this is a function?
Sara: Well, I guess I would assume that if they thought graph of ellipse is a function, I guess I would assume that they don’t understand the vertical line test thing. [Long Pause] But then the reason they would say this \( x^2 + y^2 = 4 \). They said that is not a function. Maybe they saw \( y^2 \); they just assume that it’s a function. [Pause] that’s the only reason I can think of. With \( y^2 \), I guess it would cause them to doubt that it’s a function, if they don’t really have a good understanding of this.

**Question 11**

Sara was asked to determine the relationship between the length of a side of a square and its area. She correctly wrote \( A = l \cdot w = l^2 \) and \( l \geq 0 \). When I asked Sara if this equation was a function, she drew the following graph of \( A = l \cdot w = l^2 \) (see Figure 4.44) to determine if it was a function. Then, she said, “Yes this is a parabola. It passes the vertical line test and is continuous.”

![Figure: 4.44: Sara’s solution of question 11](image)

Sara correctly wrote the equation \( (A = l \cdot w = l^2) \) that represents the relationship between the length of a side of a square and its area. She said this was a parabolic function.
The following excerpt clearly indicates her tendency to graph relations and use the vertical line test instead of algebraic or numerical approach.

Interviewer: What about the relationship between the area and length? Is that a function?
Sara: Well, just this formula would give you the full parabola, but because I guess the domain would be greater than or equal to 0 \( l \geq 0 \). It would have to be included in this. So, we only have half the parabola and. Assume the domain is only including from zero and on; then it is a function because if it included the part that well, then if it didn’t include this part then it would be a full parabola. It would still be a function, but [Pause] it’s a function.
Interviewer: Do you know what kind of function?
Sara: Well, it’s parabolic. I don’t know there is some special thing being half the parabola. In fact what you call it. It’s parabolic function.

**Question 12**

In the first part of question 12, Sara was asked to give a definition of a function. She wrote, “[Function] does not pass through the same x coordinate twice (vertical line test). [Function] has no breaks.” Her definition was very narrow. For the alternative definition of a function, she wrote,

Show the relationship between two measurable things such as time and distance traveled. There can be no breaks because time cannot stop; we are measuring something that is continuously changing and we cannot pass through the same x coordinate at different places because, using our example, you cannot be at two places at once. Look at this graph [see Figure 4.45a]. Assume we are measuring how far you have run over a certain amount of time. Can you be at 3 meters and 1 meter at the same time? No. So a graph that looks like this [see Figure 4.45b] is not a function. But this [see Figure 4.34bb] can be a function. You could have been slowing down and speeding up, but you did not mysteriously appear in two places at one time.

![Figure 4.45: Sara’s graphs for question 12](image)
As it can be seen in the excerpt above, Sara was not aware of the univalence property of functions. She gave the vertical line test without explaining why this rule works. In addition, Sara admitted that all functions must be continuous. She failed to see that functions could be discontinuous. Sara did not refer to the arbitrary nature of the two sets. That is, “functions do not have to be defined on any specific sets of objects; in particular, the sets do not have to be sets of numbers” (Even, 1990, p. 96).

**Question 13**

If you substitute 1 for $x$ in $ax^2 + bx + c$ (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

Figure 4.46: Question 13

Sara was asked to find the real solutions of a quadratic equation (see Figure 4.46). The equation $ax^2 + bx + c$ is a quadratic function where $a$, $b$ and $c$ were real numbers with $a \neq 0$. The parabola would open upwards if $a > 0$ or downward if $a < 0$. If “$a$” is positive or negative, when $x=1$, it has positive value (y value) above the x-axis. So there is a real solution on x-axis. When $x=6$, it has negative value (y value). It goes below and above x-axis. Therefore, there would be two real solutions.

Sara’s answer on the paper was, “If the discriminant is positive there are two real solutions. If negative, there are no real solutions. If zero, there is one real solution.” Her answer indicates that Sara knew a quadratic function could have at maximum two real solutions. The following excerpt explains her reasoning,

The discriminant positive, you do the quadratic equation \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \), you have square root of positive number. If this [discriminant] were zero, there would be only one solution because there would be $\frac{-b \pm 0}{2a}$. If it [discriminant] is negative, there is no real solution because it’s going to be an imaginary number. Yes, positive. It’s going to give you two numbers.

As it can be seen in the excerpt above, Sara did not try to solve this question using another representation (Even, 1989). In her response, Sara wrote these inequalities $a + b + c > 0$ and $36a + 6b + c < 0$. During the interview, she did not know or try to incorporate the coefficient
“a” to find the real solutions of quadratic equation. Sara did not try to look at another representation of the question (Even, 1989).

**Question 14**

A student is asked to give an example of a graph of a function that passes through points A and B (see Figure 4.47a). The student gives the following answer (see Figure 4.47b). When asked if there is another answer the student says: “No.” If you think the student is right-explain why. If you think the student is wrong-how many functions?

**Figure 4.47: Question 14**

In question 14, Sara wrote, “The student is wrong. The graph of a hyperbola, parabola or an infinite number of other functions can be plotted between these two points.” Sara was asked how many functions would satisfy this condition; she stated that there were infinitely many functions that could be plotted thorough points A and B.

It’s because you can have functions that get through these points, and then these two points, may be even change it little bit. You know like, down here, going off, you can increase go up there. You know you could change it in number of ways.
In her response, she drew a curve that passed through two points A and B (see Figure 4.48). However, Sara did not refer to the arbitrary nature of the two sets on which the function is defined (Even, 1989).

**Figure 4.48: Sara’s graph for question 14**

**Question 15**

Sara was asked to explain how functions and equations were related (question 15). My goal was to examine her knowledge of relationship between functions and equations. She wrote, “They [functions and equations] are all in the x, y plane and go through points A and B. They are also continuous and pass the vertical line test.” When I asked her to explain the relationship between the equation and function, she responded that equation was a way to represent functions and non-functions.

So like if you have function \( f(x) = x + 3 \), something like that. So that an equation or function written in the format of equation. But you could also draw functions as a graph or you could do a set of points instead of writing it as an equation. So those are ways that there are different [Long Pause] yes, functions can be expressed as an equation. An equation, you can represent other things that aren’t functions with an equation. Like, an equation of a circle. As you know that’s not a function, but you can still write it as an equation. Equations and graphs and tables are all like ways to represent functions and non-functions.

Sara thought that the equation was a way to represent functions and non-functions. She did not think of the arbitrary nature of functions (Even, 1989); that is, arbitrary objects that cannot be described by an equation are considered as a function. She assumed that functions must have equations even though she was aware of different representations of functions. She
said, “Equations and graphs and tables are all like ways to represent functions and non-functions.” This shows her limited understanding of functions.

**Question 16**

This is the graph of the function \( f(x) = ax^2 + bx + c \). State whether \( a \), \( b \), and \( c \) are positive, negative or zero. Explain your decision.

Figure 4.49: Question 16

The purpose of question 16 was to examine how Sara would use the shape of the graph to determine the signs of the coefficients “\( a \)”, “\( b \)”, and “\( c \)” or vice versa (see Figure 4.49). The relationship between the signs of the coefficients is very simple: “\( a \)” is negative (\( a < 0 \)) if and only if the parabola opens downward. The x coordinate of the vertex is \(-\frac{b}{2a}\) and the y coordinate which is the function value \( f\left(-\frac{b}{2a}\right)\). Since “\( a \)” is negative, the function value of \( y \) is maximum (positive); “\( b \)” and “\( c \)” are positive if and only if the value of \( y \) coordinate of the function value (i.e., \(-\frac{b}{2a}\)) is positive. Another way Sara could have used derivatives was to determine the sign of “\( b \)”. The first derivative of this function would give \( f'(x) = 2ax + b \).

When the equation \( 2ax + b = 0 \) is solved for \( x \), the \( x \) value (i.e., \(-\frac{b}{2a}\)) would be the vertex of the parabola. Since the parabola opens downward, “\( a \)” has to be negative (i.e., \( a < 0 \)). The \( x \) coordinate of the vertex \((-\frac{b}{2a})\) can be used to determine the sign of “\( b \)”. Since \( x \) coordinate of the vertex is positive and “\( a \)” is negative, “\( b \)” has to be positive to have a positive \( x \) coordinate.

Sara was sure that “\( a \)” was negative because the graph of quadratic function opens downward. She thought “\( b \)” would change the horizontal shift. Since the shift was in a positive direction, Sara believed “\( b \)” would be negative.

(2) to know if “\( a \)” is positive or negative, observe the function \( f(x) = ax^2 \) versus \( f(x) = x^2 \). “\( a \)” is negative because it determines whether the graph is concave or convex.
See (2) for method. Do the same thing for “b”. I don’t know, I am little confused, but I am going to say that “b” is negative because I think “b” changes the horizontal shift, and it seems like when something is shifted in the positive direction the coefficient is negative. “c” is positive. It affects the vertical shift.

During the interview, Sara changed her answer for the sign of “b”. She thought the sign of “b” would be either positive or negative. Sara did not notice that the value of “b” would not be equal to value of x (real solutions). If the quadratic equation is factored out, each factor is set and equal to zero. Each factor is cross-multiplied, and then added. The result of this would be equal to the value of “b”. From the quadratic equation, the relationship between the value of “b” and real solutions can be seen.

The equation is equal to zero. If you factor the equation, you have zero equals x plus number. This would be “b”. Well I guess I would say either zero or negative because it looks sort of wide.

Sara knew that the parabola would open downward if and only if “a” was negative. I asked Sara to explain how she found the sign of “c”. She correctly determined the sign of “c”. However, her reasoning was not correct. She thought that when zero was substituted for x, “c” would be the only value and the vertex (or the highest point) of the parabola. Although the value of “c” determines only the y-intercept, Sara thought that the value of “c” (y intercept) was the vertex of the parabola. Therefore, “c” was positive.

Interviewer: How did you find “c” is positive?
Sara: For “c”, kind of did it in my head. I plugged in zero for x, and then I ended up with $y = c$. Since it’s positive, it raises the vertex up above the x-axes.

Sara did not recognize the relationship between the signs of the coefficients “a”, “b”, and “c”. When the value of “a” is negative, the function value ($-\frac{b}{2a}$) is positive and maximum. She did not think of using the function value to find value of “b”.

**Question 17**

In question 17 (see Figure 4.50), Sara was asked to interpret a student’s answer.
A student was asked to find the equation of a line that goes through A and the origin O.

She said: “Well I can use the line y=x as a reference line. The slope of line AO should be about twice the slope of the line y=x, which is 1. So the slope of the line AO is about 2, and the equation is about y=2x, let’s say y=1.9x. What do you think the student had in mind? Is she right? Explain.

Sara wrote, “I think the student did a good job of estimating. I do not understand what you want out of this question. You already told me what she had in mind. She used a line as reference that she already knew the slope of. Then she compared the slope of her line to the reference.”

During the interview, Sara explained why the student could not find the equation correctly.

May be because the 45 degree angle. This angle looks a little more than 45 degree angle. So that may be where she got it from and then it’s less then twice so that’s where she get .9 from. That could be it. She was looking at the angle measures (see Figure 4.51).

As can be seen from the excerpt above, Sara thought the student used angle measures. The student utilized 45-degree angle as a reference line.
Estimation can serve as a description of the student’s work. However, this general definition is not satisfying. The student should be encouraged to calculate the exact value of slope by comparing the y value by the x value (Even, 1989). Sara knew that the student’s estimation was not correct and she suggested using a ruler to find the slope. At this point, I wanted to know how Sara could guide the student in order to find out her approach as a teacher. She said, “If you have a ruler, you could measure it or label it. You can measure it to get the slope.”

Sara’s approach was appropriate, but teacher-centered. Instead of teaching how to solve the question or saying that the student’s answer was incorrect, she preferred to create scales and measure x and y values to find the slope. I believe she had a good understanding of the problem and this allowed her to come up with a strategy. However, she could not help the student estimate the slope better. She did not say specifically what the correct answer was, how she was going to help the student, and what the student needed to do.

**QUESTION 18**

The graph of $y = \frac{1}{x^2-1}$ is not continuous at 1 and -1. The limits where x approaches 1 from left and right, $(\lim_{x \to 1^-} f(x) = \infty)$ and $(\lim_{x \to 1^+} f(x) = -\infty)$, as well as limits where x approaches -1 from left and right, $(\lim_{x \to -1^-} f(x) = \infty)$ and $(\lim_{x \to -1^+} f(x) = -\infty)$ should be used to draw the graph of this function. Substituting values for and plotting those points on the coordinate plane does not give the graph of the function (Even, 1989).
Sara was asked to explain to a student in algebra 2 how to sketch the graph of \( y = \frac{1}{x^2 - 1} \).

She suggested two methods on the paper, which could be called the point-wise approach and use of asymptotes undefined points. She said,

Finding the points for the graph for solving this problem. Another way would be to use the graphing calculator to graph this function to see what the function would look like. Still, I believed that graphing each point would be much more appropriate.

Sara created a table values for \( x \) and \( y \), and then drew the graph of \( y = \frac{1}{x^2 - 1} \) (see Figure 52). Sara knew that the function is undefined at 1 and -1. She drew these asymptotes (i.e., \( x = 1 \) and \( x = -1 \)) vertically on the graph. Then, Sara chose \( x \) values that were close to asymptotes. When I asked her to explain why she thought this method would be appropriate,

\[
\lim_{x \to \pm 1} \frac{1}{x^2 - 1} = 0.
\]

Even though Sara took advanced mathematics classes up to calculus I, II, and III, she did not attempt to use limits to determine how the function behaved when \( x \) was approaching 1, -1, infinity, and negative infinity. She only found the vertical asymptotes \( (x = \pm 1) \). She did not find the horizontal asymptotes when \( \lim_{x \to \pm \infty} \frac{1}{x^2 - 1} = 0 \). Graphing and teaching of the student how to
graph functions cannot be relied on substituting values for \( x \) and plotting those points, or using asymptotes.

**Question 19**

The exponential functions (i.e., \( y = a^x \), \( a \neq 1 \) and \( a > 0 \)) are the inverse of logarithmic functions (i.e., \( y = \log_a x \), \( a > 0 \), \( a \neq 1 \), and \( x > 0 \)), which are not employed very often. The following procedure shows how to find the inverse function of \( f(x) = 10^x \) (see Figure 4.53).

\[
\begin{align*}
y = 10^x & \text{ Interchange variables} \\
x = 10^y & \\
\log x = \log 10^y & \\
\log x = y \log 10 & \\
y = f^{-1}(x) = \log x &
\end{align*}
\]

Figure 4.53: Solution of Question 19

Sara was asked to determine whether or not log function and root function were inverse of \( f(x) = 10^x \). She wrote on the paper following algorithm (see Figure 4.54),

\[
\begin{align*}
\log f(x) & = \log 10^x = x \\
\log f(x) & = x
\end{align*}
\]

Figure 4.54: Sara’s solution of Question 19

Sara did not remember what inverse function was or how it should be presented. She said,

I am not really sure. Inverse functions [Pause] this is just a root function I think. The root function, log function are inverse [Long Pause] I don’t really know. I am trying to remember what inverse function is [Long Pause]. Like I don’t know if that would get you inverse function. Then this [see Figure 4.54] doesn’t help you out. [Long Pause] I don’t really know what to do.

As can be seen in her solution (see Figure 4.54), Sara knew that she needed to take the logarithm of both sides. However, she did not how to use algorithm to find the inverse of \( f(x) = 10^x \).

When I asked her to explain whether or not the function \( f(x) = 10^x \) had two inverses, she said,
“I don’t know that’s what I am trying to do. I don’t remember what inverse function is and how to find it. I don’t really know what I am looking for.”

**Summary of Function Questionnaire**

In this section, I summarized Sara’s responses and work in the function questionnaire to describe her subject matter knowledge for teaching functions under six aspects (Even, 1989): *essential features, different representations, the strength of the concept, basic repertoire, alternative ways of approaching, and different kinds of knowledge and understanding of function and mathematics*. In addition, I utilized an additional aspect, *analysis of students’ mistakes*, to describe Sara’s pedagogical content knowledge of the concept of function.

**Essential features**

This aspect refers to the correspondence between the arbitrary nature and univalence of functions. If one does not explain the univalence property well, mathematics might look like an arbitrary collection of rules and definitions. In the case of Sara, she was not familiar with univalence property and its use as a criterion for telling whether relation was a function.

In questions 1 and 2, she gave an example of function and non-function. In question 1, she wrote an equation for an example of function. She said the line was a function. In question 2, she wrote an equation and drew the graph of the equation. She utilized the vertical line test to show the graph did not represent a mathematical function.

In questions 3 through 10, relations were given in different representations (e.g., graph, equation, set notation). She tried to find out what the graph of the equation looked like so that she could use the vertical line test to determine whether the relations was a function or not.

In question 12, she defined a function, saying, “If graph passes through a $x$ coordinate more then once it is not a function.” As an alternative definition of a function, Sara said that “[function] has no breaks (continuous).” Both her definition of a function and her responses in the interviews revealed that Sara did not refer to the arbitrary nature of functions. The interview also revealed her excessive use of the vertical line test without explaining what it meant to fail the test, why it worked or why it was or was not a function.

**Different representations of functions**

This is about recognizing the same idea in different representations, manipulation of the idea within a given representation, and translation of the idea from one representation to another
(Even, 1989). Even (1989, p. 127) says, “flexibility in moving from one representation to another allows one to see rich relationships, to develop a better conceptual understanding and strengthen the ability to solve problems.”

Question 11 is about the relationship between the length of a side of a square and its area. Sara successfully translated the question written verbally into an equation and represented the relationship using an equation \( A = l \cdot w = l^2 \). In question 13, she was asked to find the real solutions of quadratic equation. She knew that a quadratic function could have maximum two real solutions. However, she was unable to determine how many solutions the quadratic equation would have. She did not think of approaching the problem graphically. For example, a quadratic equation can have two solutions at most, depending on the value of the discriminant. If you approach the problem graphically, a parabola can have maximum two x intercepts.

In question 16, Sara was given a parabola and asked to determine whether or not “a”, “b”, and “c” were positive, negative, or zero. The quadratic function is one of the fundamental functions in the high school curriculum and prospective teachers have been taught and have used this function during their high school and college education. At this level, they should be able to use different representations of quadratic functions. Sara failed to make a connection between graphic and algebraic representations (or translate from one representation to another) and was not able to find the signs of the coefficients. Even though she took advanced mathematics classes up to calculus III, she did not think of utilizing derivation (i.e., the maximum of the parabola) to determine the signs of “a” and “b”.

In question 18, Sara was given the equation of a rational function \( y = \frac{1}{x^2 - 1} \) and asked to draw its graph. In her response, Sara made a table (i.e., plotting points) and found the asymptotes to draw the graph of this function. She only found the Overall; she did not seem to have a good understanding of different representations of functions (e.g., quadratic, rational). She had difficulty making a connection between representations (e.g., graphical and equation) as well as translating one representation to another (e.g., equation to graphical).

**The strength of the concept**

This aspect includes teachers’ understanding of important characteristics of the concept. Even (1990) states that understanding of the concept of function must involve an understanding of the structure of functions and inverse functions. Question 19 was about inverse functions.
For example, Sara was asked to determine whether or not $f(x) = 10^x$ had two different inverse functions. She found the inverse of the exponential function by taking the logarithm of both sides. However, Sara did not know what an inverse function was nor how to find it.

**Basic Repertoire**

*Basic repertoire* includes powerful examples of basic functions such as linear, quadratic, polynomial, exponential, logarithmic, and rational. Even (1990) says, “the basic repertoire should be well known and familiar in order to be readily available.

Sara was given linear functions in equation and graphical forms in questions 3 and 8 respectively, and quadratic functions in equation and graphical form in questions 4 and 10. She recognized the functions and determined whether or not they were functions.

In question 17, she analyzed the student’s incorrect solution about the slope of a linear function. Sara knew that the second line was steeper and greater than the slope of $y = x$, so she said the student used the 45-degree angle as a reference line. The slope was not correct and could be estimated by measuring the x and y coordinates of the point A. However, she did not say how the student should choose the scale or the length of one unit. This clearly indicates that Sara’s idea to help the student was good but she could not come up with a specific solution strategy to solve the problem.

In question 13, when Sara was asked to solve a quadratic equation, she failed to use fundamental properties of quadratic functions. She did not think of using the discriminant or of approaching the problem graphically to find the maximum number of solutions. Similarly, Sara could not use the relationship between the quadratic equation and its graph in question 16. She was able to recognize the equations and graphs of the quadratic functions in the questionnaire but her understanding was not deep and conceptual.

When Sara was given the question 19 about inverse functions, she failed to use the general knowledge about exponential and logarithmic functions. She was unable to use some basic theorems, properties or graphs to solve the problem. Sara did not know what inverse function was. Since she could not solve the problem algebraically, she was unable to approach the problem graphically and showed that logarithmic and exponential functions were inverses of each other or showed that the root function was not the inverse of exponential function. For the question 18, she utilized a graph of a similar rational function to draw the graph of $y = \frac{1}{x^2 - 1}$. 

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She graphed the function by substituting values for x and plotting those points, and using vertical asymptotes.

In question 14, when Sara was explaining why there were infinitely many functions going through the points A and B, she was mentioning hyperbola or parabola. In other words, she was thinking of smooth and continuous functions. She could have drawn the graph of important functions from a high school curriculum (linear, quadratic, polynomial, exponential, logarithmic, and rational) as well as piece-wise or discontinuous functions. Her misconception about functions surfaced during the interview when Sara was answering question 15. She said, “They [functions and equations] are all in the x, y plane and go through points A and B. They are also continuous and pass the vertical line test.” A teacher should have a good understanding of these basic functions as well as a repertoire of these basic functions. This also allows her to come up with alternative ways of representing examples to students. Otherwise, a teacher might have difficulty finding an appropriate example to teach a concept or to solve a problem. Sara was able to recognize the graphs or equations of the functions. However, she had difficulty approaching the problems using alternative ways due to her limited representational repertoire and weak subject matter knowledge.

**Alternative ways of approaching**

This aspect includes different ways of approaching functions (e.g., point-wise, interval-wise, global). Even though Sara has taken advanced mathematics classes, she failed to use limits or a global approach to draw the graph of \( y = \frac{1}{x^2 - 1} \) in question 18. She should have determined the behavior of the function as x approaches \( \mp \infty \) and \( \mp 1 \). When Sara took the function questionnaire, she said that graphing would good be a good practice for the student. The student would make a table (i.e., point-wise approach) and then find the asymptotes.

**Different kinds of knowledge and understanding of function and mathematics**

The sixth aspect consists of conceptual and procedural knowledge of functions as well as meanings and understanding. I observed that Sara’s understanding of functions was procedural and rule oriented. For example, when she was trying to find the inverse of the exponential function (see Question 19 of Function Questionnaire), she was only relying on her procedural knowledge. As a result, she thought that taking the logarithm was related to inverse function. Since she could not remember the theorem for inverse functions or the algebraic rule for
In questions 13 and 16, Sara was unable to solve the problem about quadratic equations because she failed to apply the rule in question 13. She did not remember the rule in question 16. This illustrates that her understanding of quadratic functions was not deep and conceptual. Additionally, her understanding of the relationships among different representations of quadratic equations (e.g., graphical, algebraic) was not strong enough to guide her to the correct answer.

In the following aspect, Sara’s pedagogical content knowledge is described. This aspect is closely related to the ones discussed above (Even, 1989).

**Analyses of student’s mistakes**

When Sara was analyzing the student’s incorrect estimation of the slope of a line in question 17, she knew that the slope was incorrect because the second line (AO) was steeper and had a greater slope than the slope of $y = x$. Sara suggested that the slope could be estimated by measuring the x and y coordinates of the point A. She said, she would measure the x and y coordinates and find the rise over run (see Figure 4.17). This illustrates that her approach for this problem would be teacher centered because she wanted to solve the problem instead of engaging the student. However, Sara did not say how she should choose the scale or the length of one unit. This clearly indicates that Sara had a good idea of how to help the student but failed to come up with a specific solution strategy to solve the problem due to her limited pedagogical content knowledge. If Sara had strong pedagogical content knowledge or had known more about the common misconceptions about students’ mistakes, she would have suggested that the student measure scale or the length of one unit. In addition, she could have proposed more appropriate strategy to help the student.

In question 14, in order to show that there were different functions going through the points A and B, Sara said she would draw the graph of hyperbola, parabola to show that there were infinite number of other functions that could be plotted between the points A and B. This is a good example, but it represents her teacher-centered pedagogical knowledge. However, she should have used piece-wise and discontinuous functions as well as smooth and continuous basic functions (e.g., lines, parabolas) to help the student create an appropriate image of a function.
Sara did not notice the student’s difficulties, recognize the sources of students’ mistakes, and exhibited weak pedagogical content knowledge about students’ misconceptions of functions.

For the first part of question 12, Sara wrote the definition of a function; “[Function] does not pass through the same x coordinate twice (vertical line test). [Function] has no breaks.” When she was asked to give an alternative definition to help the student understand a definition of a function, she wrote, “Show the relationship between two measurable things such as time and distance traveled. There can be no breaks because time cannot stop; we are measuring something that is continuously changing and we cannot pass through the same x coordinate at different places.”

Sara admitted that all functions must be continuous. She failed to see that functions could be discontinuous. It became clear to me that Sara was not taking the student’s possible difficulties or misconceptions about functions into account when she was suggesting these examples. For example, giving an example of non-function can be very helpful, if you give different functions (e.g., continuous, discontinuous, piece-wise function). If you want to use the vertical line test, you should explain why the test works and what the purpose is. One should not use the vertical line test as an explanation of what a function is. A teacher should know or anticipate sources of students’ common mistakes and take them into consideration when one makes instructional decisions. Otherwise, students may have a different understanding of a function if you give typical examples of functions and utilize the vertical line test as an explanation of what a function is.

**Card Sorting Activity**

**Question 1**

Consider cards 1, 12 and 27. What is the easiest way to sort these cards into two piles? What criterion would be used?

---

**1**

---

**12**

---

**27.**

The Art Museum has a bulletin showing its weekly hours:

- **Monday:** Closed
- **Tuesday-Friday:** 9:00am-6pm
- **Saturday:** 10am-6pm
- **Sunday:** Noon-5pm

Graph the number of hours the museum is open for each day in March.

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**Figure 4.55: Question 1**

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The function card 1 and 12 are given in graphical representations. The function on card 27 is written in verbal representation. Function card 1 is exponential. The function of the cards 12 and 27 are periodic. In question 1, Sara was to sort the cards 1, 12 and 27 into two groups (see Figure 4.55). She wrote that function cards 1 and 12 were graphical representations and that function card 27 was a verbal representation. She grouped the function cards according to their representations. During the interview, Sara was asked whether or not the function cards 1, 12 and 27 could be sorted differently. The following excerpt explains how she categorized the function cards 1, 12, and 27.

Sara: This one [27] is I think periodic.
Interviewer: How do you know that 27 is periodic?
Sara: What is going through is the time [Pause] time is showing that the art museum is open throughout the week. That week is repeating. Tuesday and Friday open at the same time. Saturday and Sunday open different times, and repeats. That’s periodic. It keeps following that pattern. “12” is periodic also because it’s following the same patterns. It’s shown. So, I would say 12 and 27 are periodic. 1 is exponential.

Sara grouped the function cards 1, 12 together and separated the card 27. She said, the function on cards 12 and 27 were periodic and the card 1 was exponential. Sara sorted the cards using two different ways. She used different representations and categories of functions to sort the cards.

Question 2

If you were to sort 28 cards into four piles, which criterion makes the sorting easiest?

Figure 4.56: Question 2

In this question, twenty-eight function cards were given in using four different representations and seven categories of functions. These function cards could be sorted using different representations and categories of functions.

In this question, Sara was asked to group 28 cards into four piles (see Figure 4.56). She said, “graphical, verbal, table, formula.” I asked Sara whether or not the 28 function cards could be sorted differently. Then she started grouping the function cards. I asked her to explain her strategy for grouping these function cards. She said, “Those are all x to the power. Having that
exponential and logarithmic together, they have similar graphs. They are inverses of each other, I think.” Then she drew the graphs of the function cards 1 and 26 to show that they would be inverses of each other (see Figure 4.57).

Well, this one [26] is log, that one [1] is exponential. If you rotate or flip this one over this like diagonal axes. Number 26 is logarithmic. Number 1 is exponential. If you were to flip this over, there would be like reflections of each other.

Figure 4.57: Sara’s graph of Function Cards 1 and 26

After drawing the graphs of the function cards 1 and 26, Sara was confused, but she was not sure how the cards were related.

So, I’m not sure, but they are, but they are related. So, that’s why I would group them together. And then you can have periodic functions. You can also have hyperbolic functions. I don’t think any other kind. So, that’s four groups.

As can be seen excerpt above, Sara failed to notice that exponential and logarithmic functions are inverses of each other.

Sara thought there were six categories of functions. She failed to recognize the polynomial function in the cards. Sara wrote the twenty-eight function cards under the category of four. These categories were \( x^n \), exponential and logarithmic, hyperbolic, and periodic.

Sara thought that the function cards 13 and 17 were hyperbolic. The following excerpt explains her thinking:

Interviewer: What are the graphs that were hyperbolic?
Sara: This one is. Number 17, 13 and I’m sure there are others, but I have to graph these, but I think there is one hyperbolic, may be this one.

At first, Sara was thinking the function card 4 was linear, and then she decided to draw the graph of card 4 (see Figure 4.58). Still, Sara was not sure whether or not the function card would be cubic. She put the card 4 under the category of x to the power.

I don’t know if you could graph it. I’m not sure. It goes. I don’t know if it’s linear. It could be linear. No, it couldn’t, but it could be like cubic maybe. Maybe $x^3 + x$. Since it’s shifted over [see Figure 4.58]. Actually, I’m not sure.

Figure 4.58: Sara’s graph of the Function Card 4

After talking about the function card 4, Sara categorized the function cards 3, 11, 5, 8, 23, 27.

Number 3, because it says she is filling a cubical container, probably will end up being like a cubic function and talks about volume. So, that [3] would go with the x to the power group. This [11] will be linear because if you have [Pause] number of gallons of gasoline, she can purchase and then the cost increases, the gallon will decrease. It should have linear relationship for number 5. That would go into that [x to the power] category. “8” should be quadratic; it also goes in that category. It’s just a constant growth, so 11 is linear. 23 is also growth rate, it increases as annual rate of 2 percent. That would an exponential function. As the population increases, 2 percent would become a larger number. 27 is periodic. Number 24, I am not sure about. I think 24 is a linear function. 16 is logarithmic. 21 is, that would be parabolic. Number 25 and 28 are exponential [Pause] yes, 28 would be exponential. 22 is to the 4th power 2, 3, 4, 5, 6, 7. I graph 14 [see Figure 4.59]. I don’t know what 24 is [see Figure 4.59].
Sara decided that the function cards 13, 14, and 17 were hyperbolic. The function on cards 10, 12 and 27 were periodic. However, she did not know what kind of function card 24 was. I asked Sara to create a table and write the twenty-eight cards under the categories of four. I wanted to see how she organized the cards clearly into four groups (see Figure 4.60).
**Question 3**

Sort cards 3, 15, and 19 into two piles. In what sense is your sorting criterion different from the criterion used in exercises 1 and 2?

<table>
<thead>
<tr>
<th>3.</th>
<th>15.</th>
<th>19.</th>
</tr>
</thead>
</table>
| Denise is filling a cubical container measuring one foot on each edge with water. She notices that it takes a lot more water when each dimension of the cube is increased. She wonders how much the volume of the cube increases when each dimension is increased $x$ units. | $y = 2x + \pi$ | ![Figure 4.61: Question 3](image)

Function card 3 is polynomial and given in verbal representation. The function on the cards 15 and 19 are linear. Function card 15 is given as equation. The function on the card 19 is given in graphical representation. In question 3, Sara was asked to sort the function cards 3, 15 and 19 (see Figure 4.61). She said, “15 and 19 are linear functions. 3 is a cubic function.”

When I asked her whether or not these cards could be sorted differently, she said, “Well, the only thing I can think of is because 19 is decreasing linear, and 15 is increasing and so the 3 is increasing. It could be grouped like that as increasing and decreasing.”

Each dimension is 1
The volume of the cubical container is $(1)^3 = 1$
Each dimension is $1 + x$
The volume of the cubical container is $(1 + x)^3$
$(1 + x)^3 = (x^2 + 2x + 1)(x + 1)$
The increase in volume $x^3 + 3x^2 + 3x$ where $x$ is the increase on each side.

![Figure 4.62: Solution of the card 3](image)

Sara did not make any attempt to find what the function card 3 was. She thought it was a cubic function. In card 3, the volume of a cubic container is $(1 + x)^3$ and a cubic function (see Figure 4.62). Function card 3 asked to find the increase in volume $(x^3 + 3x^2 + 3x)$ where $x$ is the increase on each side. The increase in volume was a polynomial function. Since Sara thought the function card 3 was cubic. She did not make any attempt to find the equation of the increase in volume.
Question 4

In what sense are cards 6, 8, and 21 alike?

<table>
<thead>
<tr>
<th>6</th>
<th>8</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image.png" alt="Graphical representation of card 6" /></td>
<td>Fred is considering which size pizza is a better buy. He wonders what happens to the area of the circular pizza when the diameter of the pizza is doubled.</td>
<td>x</td>
</tr>
<tr>
<td>$y$</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Figure 4.63: Question 4

Function card 6 is quadratic and given in graphical representation. The function on card 8 was quadratic and written in verbal representation. Function card 21 is also quadratic and given in table representation. In this question, Sara was asked to determine how these cards were alike (see Figure 4.63). She wrote, “They are all periodic.” During the interview, I asked Sara to explain how she knew the function cards were quadratic,

Interviewer: How did you know these were quadratic?
Sara: For 21 because it’s symmetric like [Pause] yes $x=-1$ is zero, and then one from either direction is $y=1$. It’s symmetric there. It continuous on because increase. So, it’s because it has that vertex in that symmetry.

As it can be seen from the excerpt above, Sara did not provide any reasoning for her answer. She did not make any attempt to find what kind of functions the cards 6, 8 and 21 were. Sara failed to see that the function cards 6, 8 and 21 were quadratic functions.
**Question 5**

Sort cards 9, 13, 17, and 22 into different piles, two different ways. Place the cards into the following 2 x 2 matrix; \( C_1 \) and \( C_2 \) represent one type of criterion and \( D_1 \) and \( D_2 \) represent a different type of criterion.

<table>
<thead>
<tr>
<th>( C_1 )</th>
<th>( C_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( D_1 )</td>
<td>( D_2 )</td>
</tr>
</tbody>
</table>

9. \[ y = 3x^5 - x^4 + 4x^3 - 10x^2 + x - 6 \]

13. \[ y = \frac{2}{3x} \]

17.

22.

![Figure 4.64: Question 5](image)

The function cards 9 and 13 are given as equations. The card 9 is a polynomial function. The function cards 13 and 17 are rational functions. The function on the card 17 is given in graphical representation. The function card 22 is trigonometric and given in graphical representation.

In question 5, Sara was asked to sort the function cards 9, 13, 17, and 22 into two piles using two different ways (see Figure 4.64). She wrote, “1st way, 17 and 22 are graphical representations. 13 and 9 are formulas. 2\textsuperscript{nd} way, 13 and 17 are hyperbolic, and 22 and 9 are to the 5\textsuperscript{th} power.”

Sara found that the function cards 13 and 17 were hyperbolic. She did not recognize that function card 9 was 5\textsuperscript{th} degree polynomial. The function on the card 22 was a trigonometric function, but Sara did not know what kind of function the card 22 was.

When I asked her whether or not these cards could be sorted differently, Sara said, “I can’t think of any other way to sort. It sort of would be the same thing instead of saying hyperbolic, you could say asymptote, it’s the same thing.” According to Sara, an alternative way of sorting these cards would be the same as her first two ways. The function cards 13 and 17 had asymptotes. 22 and 9 were to the 5\textsuperscript{th} power. I asked Sara how she knew the function cards 13 and 17 had asymptotes.
I know, hyperbolic have asymptotes anyway. I just know that. From this graph [card 17], you can pretty much assume because it looks like it has an asymptote. So, from the graph it’s hard to tell. From the formula, it’s easier to tell because when $x$ is equal to zero $y$ is undefined. So, you know that $x = 0$ is going to be an asymptote, because the function cannot pass through that line, and I guess if you rearrange this formula to have $x$ equals whatever, it is going to be $x = \frac{2}{3y}$ then; it would work the same way when $y$ is zero, $x$ couldn’t be defined. So, that’s why there is an asymptote.

As it can be see from the excerpt above, Sara admitted that it was easier to find the asymptotes from than equation then the graph. Even though Sara knew the cards 13 and 17 had asymptotes, but she classified the cards 3 and 17 as hyperbolic.

**Question 6**

Sort cards 1, 6, 12, 16, 17, 21 25, and 28 into two piles. Describe carefully the criterion you used to sort the cards.

Figure 4.65: Question 6

Sara was asked to sort the function cards 1, 6, 12, 16, 17, 21 25, and 28 into two piles (see Figure 4.65). She wrote, “1, 6, 12, 17 are graphical representations. 16, 21, 25, and 28 are table representations.” When I asked Sara to determine whether or not the function cards could be sorted differently,

By looking at these four graphs, none of them are in the same category. They are all different types of graphs, types of functions. So, you couldn’t make two categories since they are different kind of functions.

Sara knew the function cards could not be placed into two piles using categories of functions (e.g., exponential, quadratic).

**Question 7**

Sort cards 1, 6, 7, 16, 21, 26 and 28 into two piles. Describe carefully the criterion you used to sort the cards.

Figure 4.66: Question 7

In question 7, Sara was asked to sort the function on cards 1, 6, 7, 16, 21, 26 and 28 into two piles (see Figure 4.66). She grouped the function cards using two different ways (see Figure 4.67).

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When I asked Sara whether or not the function cards could be sorted differently, she said, “I don’t see other way. I don’t see anything.” As can be seen from Sara’s solution (see Figure 4.67), she had a good understanding of three categories of functions and the use of their representations.

**Summary of card sorting activity**

In question 1, Sara recognized the function cards 1, 12 and 27, but he was able to sort them both using their representations (graph and verbal) and categories of functions.

In question 2, I asked Sara to sort 28 cards into four piles. She sorted the function card into four groups according to their representations. Even though she did not recognize some functions in the activity, she sorted the cards using four categories ($x^n$, exponential and logarithmic, hyperbolic, and periodic).

In question 3, Sara recognized the linear functions given as equation and graph (the card 15 and 19). She thought of the function card 3 as cubic, even though this card was polynomial function and represented the relationship between the increase in volume and increase in the dimension of the cubic container. However, Sara did not make any attempt to find the equation of the increase in volume.

In question 4, three quadratic functions (the cards 6, 8 and 21) were given in graphical, verbal and table representations. Sara was asked to determine how these cards were alike. However, Sara thought of the function cards as periodic. She failed to recognize that these function cards were quadratic. At this point, it became clear to me that Sara did not understand quadratic functions well.

In question 5, Sara was asked to sort the function on cards 9, 13, 17 and 22 into two piles using two different ways. She sorted the function cards according to their representations (equation and graph). As a second criterion, Sara used different types of functions. She said 13
and 17 were hyperbolic, 22 and 9 were 5th power. However, Sara failed to see that function the function card 9 was polynomial. The function card 22 was trigonometric.

In question 6, Sara again sorted the function cards according to their representations. She could not sort them according to their types.

In question 7, Sara sorted the functions cards into two piles using two different ways. In her response, she grouped the function cards using their types (i.e., exponential, quadratic, logarithmic) and different representations (graph, equation, verbal).

Sara sorted them correctly according to their types. She was unable to make a connection between the exponential and logarithmic functions. Her strategy of determining the types of functions relied on plotting points and matching them to familiar functions. Overall, she sorted the function cards according to their representations. Her understanding of basic functions (quadratic, linear, exponential) was weak and limited. She seemed to understand different representations of quadratic functions. However, in the function questionnaire Sara was unable to make connections among different representations (e.g., graph, algebraic) to determine the signs of a, b, c or to solve the quadratic equation. Similarly, in the card sorting activity, I noticed Sara did not recognize the quadratic function. In the function questionnaire as well as in card sorting activity, Sara failed to find the inverses. She did not know what inverse function was and how she could find it. In the card sorting activity, Sara knew exponential and logarithmic functions were related, but she failed to notice that these two functions were inverses of each other. At this point, it became clear to me that Sara’s understanding of logarithmic and exponential function and relationship between these two is very weak. In addition, her understanding of the relationships among these representations is fragile and limited. As a result, different representations of these functions are separate and not connected.

**Analysis of Lesson Plans and Their Teaching Episode**

**Sara’s analysis of two lesson plans and their teaching**

The description of these lesson plans as well as their teaching videos are provided at the beginning of this chapter.

The first lesson (or lesson plan) focused on *Theoretical and Experimental Probability* (see Appendix C) and the second lesson (or lesson plan) focused on *Fundamental Counting Principles* (see Appendix D). Sara was asked to determine which of these lesson plans she liked better and to explain her reasoning. The following excerpt illustrates her thinking:
I like the second one better. Just because of the way he pulled things together. It was very nice and I think even he got to one point where he kind of like, he gave them one activity with different kinds of girlfriends, and there were so many way to do it. He was like you see there is a reason we need this other method; some of the students were like aha! It was just really nice how it worked out.

I asked Sara why she did not like the first lesson. She found the first lesson less organized and structured because Mr. A did not prepare the first lesson prior to his teaching. As he was teaching, Mr. A decided what to include into the lesson.

He [Mr. A] had like some good activities. It just didn’t have the same flow and structure as the second one. Like it didn’t seem as intentional and used some examples to show them [Pause] how to show them probability. He just did some different examples and sort of little discussion but it was less organized I guess. On his reflection, he said he had not prepared for that lesson. Second day, he had prepared for the lesson. So first day, he was just coming up with things. Spear the moment and wasn’t really prepared for what the students were going to say and they’re just. He didn’t know what he was going to do. So, it was less organized.

Sara was asked to explain why she liked the video teaching of the second lesson (good) better then the first lesson (bad). She liked how Mr. A had the students solve the first example with different methods in the second lesson (good). When Sara compared these two lessons, she could not explain why the video teaching of the first lesson was not better. The following excerpt explains her thinking;

Sara: [Pause] what he did really [Pause] second one he had this first example and discussed the ways to go about. All the different possibilities you can come up with, and then so they come up with this tree model. The second one, he used a tree diagram, and then [Pause] I just like that he built off this first example and then used the second example to transition into this final point I guess, and then first one, I remember he had an example with where all the students had the flip chips and [Pause] I just like how this one [second lesson] flows.

Interviewer: What you mean with ‘flow’?
Sara: [Pause] the lesson different examples built of one-another, transition to the different points in it. It’s just nice to move the class more smoothly. The students were able to see where he was going with this. When he got to choose the next point and the first one [first lesson] wasn’t like that have that intentional smooth movement.

Sara only recognized that the first lesson (bad) was prepared prior to teaching of this lesson. She said, “Well he was prepared. Preparation and thought went into the second one. The first one, he wasn’t able to make it. Keep control of the class as well [Pause].” However, Sara thought that Mr. A achieved his goals for the first lesson (bad). She said, “He discussed.
He compared *Experimental and Theoretical Probability* because that was experimental and he had them all come up with random numbers, I remember. It seems like a good activity to help them learn it.”

Sara thought that the problems were very motivating in the second plan. She recognized that Mr. A used these problems to guide the students. In the class, one of the groups came up with tree diagram and showed their solution at the board. Then, Mr. A told the students this was called tree diagram. Sara failed to see that Mr. A encouraged students to look for different solutions. He explained *Fundamental Counting Principles* to the students. He helped students to develop their understanding of fundamental counting principles. For the first lesson (bad), Sara thought the first lesson’s problems was not very motivating, but she could not explain or give any reasoning.

First problem was simple. He used that to help them to understand how to make one of these trees, diagrams. The second one was really complicated, and they got frustrated with it. He used that frustration to show them that [Pause] you need another. You need a better way to go about it. He showed them how to make tree diagram, which they need to know how to do. It’s sort of the basic foundation for what they went onto learn. He used them to make them frustrated. They would realize that for motivation for them to need to know a better way of solving these problems. So, I guess that he used that motivation, and this one [first lesson] that wasn’t really a motivation. They just did an example on.

She thought the chip activity did not help students to learn what probability was. According to Sara, the activity did not go any further then flipping the chip because the students was not taught about the probability. The following excerpt explains her thinking about the first lesson,

This coin [chips] flipping did help them. I think they got little distractive worrying about flipping and keeping track of the numbers. Even though they are doing an experiment to trying to figure out, if you do not teach them about the probability. Like they are not really thinking about probability at the time they get, they are just thinking about flipping the points. So, and then he made them focused, put all the data up on the board, and discussed it. I don’t know, for some reason it doesn’t seem like it.

As can be seen excerpt above, Sara thought the first lesson did not help students to learn what probability was. She thought that there was a discussion about the activity. Mr. A teacher wrote the students’ data that included a number of yellow and red chips. Then, he used the formula (i.e., number of possible outcomes divided by all number of outcomes) to find the
probability. Sara failed to see that Mr. A did not let students discuss their findings. Sara could not explain why she thought the first lesson was not better then the second lesson.

Researcher’s observation about Sara’s analysis of the two lesson plans and teaching of the lessons

Sara recognized the good and bad aspects of these lessons. She saw that the first lesson did not help students to learn what probability was. She found the lesson was not organized because the lesson was not prepared prior to teaching. Sara recognized that Mr. A gave the definitions, formulas and solution of the problems. She thought there was a discussion about the problems. Sara failed to see that Mr. A explained what needed to be done in class activities and gave the solution to them. Therefore, there was not any discussion about students’ findings. For the second lesson, she thought the lesson was built on problems that Mr. A was able to make transition from one problem to another. Mr. A did not directly tell the students answers to the problems. He let students investigate the questions and discuss their answers.

Preparation and Analysis of The Lesson Plan on Exponential Function and Teaching Episode

Summary of Sara’s lesson plan on exponential function

Sara was given a lesson plan guideline that included objectives of the lesson (see Appendix E). This lesson plan was prepared to teach a small group of senior high school students at the Southern High School (see Appendix G). By the end of the lesson, the high school seniors would be able to teach graphing exponential functions to the students. In this lesson, the activity called “money example” would be employed. The following excerpt explains how she planned to introduce the activity:

Now I am going to do a little experiment. You can probably gather from the fact that I want to be a teacher that I don’t care that much about money! However, as we all know, it is a common goal in our American culture to want to get as much money as possible. So what I want to know is this: are you guys typical Americans or are you like me?

After her explanation, Sara would write three exponential functions, \( f(x) = 10^x \), \( f(x) = \left(\frac{4}{5}\right)^x \) and \( f(x) = (-2)^x \) on the board. She would explain that x-axis would represent time and y-axis would measure dollars. The students would be asked what the graph of these functions would look like. Sara would ask students to graph the functions. After the students completed graphing
the functions, she would ask them to explain how they found it. Sara would ask them to draw
the graph of functions on the board. Then, she would ask the students who would want which
one of these three exponential functions, \( f(x) = 10^x \), \( f(x) = (\frac{4}{5})^x \) or \( f(x) = (-2)^x \). After the
students picked their choice of functions based on their graphs, she would tell the students that
this class was full of typical Americans. At the end of the lesson, Sara planned to review what
the students learned in this lesson. They would be asked the following questions, “What have
we been learning about it today? How do you plot the graph of an exponential function? What
are exponential functions used for? What do they represent?”

**Analysis of Sara’s lesson plan**

In this section, I wanted to provide my comments on Sara’s lesson plan on exponential
functions. I asked Sara what she thought about writing a lesson plan. She told me that she
started taking short notes about her ideas. Since one of two objectives focus on teaching how to
graph exponential functions, Sara chose equations for this lesson. She would ask students to
graph the equation of exponential functions. She thought this activity would be a good start for
students to learn about exponential behavior. It would help them identify data that displays
exponential functions.

Well, I just started brainstorming, really. Just started writing notes, things that I could do.
Writing down what I wanted to say. I started out with just getting them to draw the
functions because that was the first objective. They could graph exponential function. I
thought I could have each student attempt to graph the function, and then discuss it
different ways that they went about it. It’s just graphing or solving for different points,
and then graphing those points. Just to have them discuss, sort of figure out a way to
graph an exponential function. I thought that would be a good start and then they’re
supposed to identify the data that displays exponential functions. Okay, well, I wanted
first just to graph it and then discuss.

Sara wanted to show three different exponential functions. However, Sara was not sure
whether or not she should choose basic exponential functions for this lesson. She also changed
her activity for this lesson.

May be I should do more basic exponential functions because I was thinking of doing all
different types, every different sort of exponential function I could think of. I wanted to
discuss what the graph would represent and that would be the identifying data that
displays exponential behavior. I was going to give the example about money [see
Appendix G], have them graph, choose which one they would represent their own. I
thought it would help them to be more interested in it.
Sara explained that she changed one of the tree exponential functions for this lesson. When I asked Sara why she changed the examples, she thought these functions would be good for students’ practice.

[Pause] well I know that at first I was going to do \((-2)^x\). I thought that was little ridiculous so I made it positive 2. So \(10^x\) increase much more quickly then \(2^x\) and then \((\frac{4}{5})^x\) would decrease, kind of even out. So I figured they would choose \(10^x\) for the greatest increase. I just wanted to give them a few functions really to have more practice with. So, I thought those would be all right functions to use.

As can be seen from the excerpt above, Sara could not explain in detail why she selected these equations for her lesson. She failed to see that \((-2)^x\) was not an exponential function.

Exponential function \(f\) given by \(f(x) = a^x\) for every \(x\) in \(R\), where \(a > 0\) or \(0 < a < 1\) and \(a \neq 0\).

I asked Sara how she found the “money example.” She thought about different examples of exponential functions in real life and created this activity.

Well I was sort of brainstorming what different things that exponential function could represent and so I have stock, population, and all sort of things. So, I just thought it up. I knew I wanted to do another example. So, I was trying to come up with something. This is what came to me. I just brainstormed on my own.

Sara talked about the graphing equations and behavior of exponential functions. She did not explain how the students would learn exponential functions without definition of exponential functions.

Sara was asked whether or not it was difficult for her to write the lesson. She thought the objectives were not specific. She explained that it was difficult to select examples due the broad definition of objectives.

I have never written any other lesson plan. The only thing was that objective was sort of broad. So, I wasn’t sure. I was saying like I had all different sorts of graphs that I wanted to use as an example. I wasn’t sure if I should use all different sources or if I should use like three basic types of exponential functions. So the students won’t get confused. They’re just starting to learn about this. So that because it’s so broad I wasn’t sure.

**Researcher’s observation about analysis of Sara’s lesson plan**

Sara was thinking of using the activity called, “money example” for this lesson. She planned to give three functions and ask students to graph it. Her goal was to teach them
increasing \((f(x) = 10^x)\) and decreasing \((f(x) = (\frac{4}{5})^x)\) exponential functions within this activity. During the interview, when Sara was asked to explain why she chose these examples, she could not give any specific reason. She had difficulty choosing examples for this lesson. She was not sure whether or not the examples were appropriate for this lesson. She could not provide appropriate reasoning for why she selected these examples. Up to this point, she did not include a way to introduce definition of an exponential function and its properties. For instance, during the interview, Sara changed her example of \((-2)^x\). She did not recognize that this was not an exponential function. Exponential function \(f\) is given by \(f(x) = a^x\) for every \(x\) in \(R\), where \(a > 0\) and \(a \neq 0\).

She did not think how these examples would help a group of high school seniors to learn the concept. She planned to have them graph the equations. She thought this would be sufficient for them to identify data that displays exponential behavior. However, Sara did not explain how she would teach the students to graph exponential functions or what kind of strategies she would use to help students learn better. Up to this point, I do not see any clear discussion of the definition of the exponential function or examples of exponential functions. For example, the exponential function is for \(y = a^x\), where \(a > 0\), \(a \neq 0\). Sara is not thinking of discussing these definitions or giving examples of these definitions in her lesson plan. In addition, I believe it would be helpful to divide exponential functions into two groups; exponential functions whose base greater then 1 \((y = a^x, a > 1)\), exponential functions whose base is between 0 and 1 \((y = a^x, 0 < a < 1)\). When the definition is discussed, the instructor makes sure that the students know the base cannot be negative numbers, zero or 1. Even though Sara did not include definitions of exponential functions in her lesson, I was hoping that she would include definition of an exponential function as well as properties of exponential functions in her teaching. Once the definitions and the two sub-types of exponential functions are discussed, the instructor should give the basic properties of an exponential function. For example, the function, \(y = 2^x\) if you coordinate several points for the graph. The list of the points would be the following sets. The set, \((-2, -1, 0, 1, 2, 3)\) is the domain of the function consisting of the values of \((-\infty, +\infty)\). The set \((\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8)\) is the range that included all possible values \(y\) for \(x\) in domain \((0, \infty)\). If you plot
the points and sketch the graph, \( y = 2^x \) is an increasing function and its graph rises (see Figure 4.68).

![Graph of \( y = 2^x \)](image)

Figure 4.68: Graph of \( y = 2^x \)

When one moves from left to the right, \( y = 2^x \) will approach to the x-axis. Thus, x-axis is horizontal asymptote. As \( x \) increases through positive values, the graph rises rapidly. The equation \( y = 2^x \) does not have x-intercept and the y-intercept is \( y = 1 \). The graph of \( y = 2^x \) is increasing on \((-\infty, +\infty)\). It is an increasing function when it comes to explaining whether or not \( y = 2^x \) is a function. The horizontal line on the Venn diagram would show this as a function. The equation \( y = 2^x \) is one-on-one can be shown using the horizontal line test. The graph of \( y = 2^x \) is not symmetric. Thus, it is neither an even nor an odd function because a function \( f \) is even if the graph of \( f \) is symmetric with respect to the y-axis. Algebraically, \( f \) is even if and only if \( f(-x) = f(x) \) for all \( x \) in the domain of \( f \). A function \( f \) is odd if the graph of \( f \) is symmetric with respect to the origin. Algebraically, \( f \) is odd if and only if \( f(-x) = -f(x) \) for all \( x \) in the domain of \( f \). As \( x \) approaches \( \infty \), \( y \) values will get bigger as \( x \) approaches \(-\infty \). “\( y \)” values (the graph of \( y = 2^x \)) will approach the x-axis but it will never touch or cross the x-axis. Then, the graph of exponential function whose base between 0 and 1 should be given and properties of the graph should be discussed as a class. For example, the equation, \( y = \left(\frac{1}{2}\right)^x \) where \( x \) is restricted to rational numbers, if one coordinates several points for the graph the list of the points would be the following sets. The set \((-2, -1, 0, 1, 2, 3)\) is the domain of the function consisting of the values of \((-\infty, +\infty)\). The set \((4, 2, 1, \frac{1}{2}, \frac{1}{4})\) is the range that included all possible values \( y \) for \( x \) in
domain \((0, \infty)\). If one plots the points and sketches the graph, \(y = \left(\frac{1}{2}\right)^x\) is a decreasing function and its graph rises (see Figure 4.69).

![Figure 4.69: Graph of \(y = \left(\frac{1}{2}\right)^x\)](image)

When one moves to the right, \(y = \left(\frac{1}{2}\right)^x\) will approach to the x-axis. When one approaches to the left \(y = \left(\frac{1}{2}\right)^x\) goes up. In terms of explaining whether or not \(y = \left(\frac{1}{2}\right)^x\) is a function, the horizontal line on the Venn diagram would show this as a function. The equation \(y = \left(\frac{1}{2}\right)^x\) is one-one can be shown using the horizontal line test. The graph of \(y = \left(\frac{1}{2}\right)^x\) is not symmetric. Thus, it is neither an even nor an odd function because a function \(f\) is even if the graph of \(f\) is symmetric with respect to the y-axis. Algebraically, \(f\) is even if and only if \(f(-x) = f(x)\) for all \(x\) in the domain of \(f\). A function \(f\) is odd if the graph of \(f\) is symmetric with respect to the origin. Algebraically, \(f\) is odd if and only if \(f(-x) = -f(x)\) for all \(x\) in the domain of \(f\). As \(x\) approaches infinity \((\infty)\), \(y\) values will get bigger as \(x\) approaches negative infinity \((-\infty)\). \(\text{\textquotedbl}y\text{\textquotedbl}\) values (the graph of \(y = \left(\frac{1}{2}\right)^x\)) will approach the x-axis but it will never touch or cross the x-axis.

Sara planned to ask students to graph the equations by plotting \(x\) and \(y\) values. Although she mentioned that she would have a discussion to explain the graphical effect about increasing and decreasing functions, I was hoping she would compare these graphs and show them how the horizontal and vertical shift occurs.

**Summary of Sara’s teaching**

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In the class, there were four female and one male high school seniors. The lesson began when Sara gave them a worksheet and told them that it had two functions. She asked them to try graphing these functions. Sara told the students, “Right now you have no idea. My hint is if you just plugged in a number for x, any number, solve to see what y would equal, points.”

One of the female students, Jennifer asked, “You want us to find what x point is and find out for the y.” Sara told Jennifer that she could find a couple of points for x and tried to graph it. Sara told the class to try any numbers. As the students worked on the questions, she stood in front of the board and looked at her notes. Then, Sara wrote the equation \( f(x) = 4^x \) on the board. After writing the equation, she walked towards the students’ desks. She walked around the students and looked at their work. She approached a female student, Niki, looked at her work. Sara told Niki, “Try numbers that would bridge.” After Sara finished talking to Niki, the male student, Mike, asked Sara what he could do with \( f(x) \). Sara told him that \( f(x) \) was the same as “\( y \)”. This was \( y = 4^x \). When x was equal to 1, y equals 4. After Sara’s response, Mike asked if this was the same as \( y = 4^x \). Sara said yes. Then, another female student, Jenny asked for help. She said, “I am lost.” Sara walked to Jenny’s desk. Jenny asked how she could graph 25. Mike told Jenny this (25) was a big number. In the meantime, Sara stood next to Jenny. She did not say anything about Jenny’s question. Another female student, Ashley, and Mike tried to explain to Jenny how she could graph it. When these three students were talking, Sara was quiet. The students helped each other. After the students’ discussion, Sara told Jenny to try negative numbers for x. Then, she walked around the students. Sara approached Niki again. Sara checked her work. She did not say anything to her. When Sara finished checking her work, she approached the last student, Sandy. Sara checked her work but she did not say anything to the student. Sara walked around the class. She stood behind the students’ seats and looked at their work. She did not say anything to them. She walked in front of their seats and stood in front of the board. She said, “I told you to graph it, but we need to make space for that. Let’s do this together.” She stood up next to the board and drew a table for x and y values. Then, she drew a Cartesian coordinate system. She asked the students what they plugged in for x. One of the students said 1. Sara wrote 1 for x value. She wrote \( y = 4^x \) on the board. Then she pointed out \( f(x) = 4^x \). She told the students, \( f(x) = 4^x \) was the same as \( y = 4^x \). Sara asked when x was equal 1, what y was equal. One of the female student said 4. She plotted the point (1, 4) using a
dot. Sara told the students to try zero for x. She wrote zero in x value. She asked when x was zero what the value of $4^0$ would be verbally. The students talked to each other about $4^0$ for a few seconds. Then, Sara told the students it was just 1. One of the female students asked Sara “did you say $4^0$? That goes to 1.” Sara told the student, “I don’t know. It just equals one. I’m not sure how to explain it. It’s confusing. I don’t know. It’s 1, basically. We’ll go with 1.” Sara moved on and plotted the points (0, 1) on the Cartesian coordinate. However, she just plotted points. She did not write the values for x and y on the Cartesian coordinate. Sara asked the students when x equals -1, what the y would be. A few students told her that it would be equal to -4. She explained to the students that if it was 4 times -1, it would be -4. Sara waited a few seconds. She started writing on the board as she explained to the students, telling the students that a number “n” to the x power ($n^x$) was equal to $n^{-x} = \frac{1}{n^x}$. She asked the students if they had seen this before. The students told her no. She went back to her question. She told the students $4^{-1}$ would be $\frac{1}{4}$. The students did not recognize the rule for negative exponents. Sara asked them again if they had seen this rule before. The students said no. Sara went back to the question. She wrote $\frac{1}{4}$ in y values. She told the students that if they did not know this rule, their graph would change. Then Sara plotted the points (-1, $\frac{1}{4}$), but she did not show the values for x and y on the Cartesian coordinate. One of the students asked what x and y values were on the Cartesian coordinate. Sara pointed it out on the graph and told the student x was -1 and y was $\frac{1}{4}$. She told the class to try for -2. She asked what they thought about $4^{-2}$. The students were quiet. Then, Sara wrote $4^{-2} = \frac{1}{4^2}$. She asked the students the value of $\frac{1}{4^2}$. The male student, Mike, told her that it would be $\frac{1}{16}$. Then, using the tables values (see Figure 4.70) Sara plotted the points (-2, $\frac{1}{16}$) on the Cartesian coordinate. She did not write the values for x and y on the coordinate. She told the students $\frac{1}{16}$ would be very close to zero. She connected the points and drew the graph of $f(x) = 4^x$. She said, “This is like what an exponential function does. It goes along like this. It shoots up really fast.”
Sara told the students to work on the second question on their worksheet and tried to graph it. One of the female students told Sara that she had an odd graph. Sara looked at her graph, but she did not say anything. Another female student asked the name of this graph. Sara told the student that this was called exponential function. She also told the students that this function always had an exponent. They were all raised to the power. Sara stood in front of the students for a minute as they worked on the second question on their worksheet. Then, she walked towards the board.

Sara told the class, “Let’s try another example.” She erased what had written on the board. Sara told the students, “Let’s do one more question and then I’ll do different types of examples.” She wrote \( y = -5^x \) on the board. She created a table for x and y values (see Figure 4.71).

Then, she wanted to look at what happened to the x. For x, one of the students said 1. Sara wrote 1 in x values. Then, she asked the students what the y value would be. The students did not answer her question. One of them asked if the graph would be different than the first graph. Sara explained to the students verbally that \( 5^1 \) would be equal to 5. \( 5 \) took the negative and it would be -5. She told the student that there would be a difference between the first and second graph. After answering the student’s question, Sara wrote \( y = (-5)^x \) on the board. She created a table for x and y values (see Figure 4.71).
Then, Sara explained to them when \( x \) was equal to 1, \( y \) would be -5. She wrote zero for \( x \) values of two equations \((y = -(5^x)\) and \(y = (-5)^x\)). She told the students that when \( x \) was zero, the value of \( y \) would be -1 for \( y = -(5^x)\). Then, she told the students when \( x \) was 0, the \( y \) value would be 1 for \( y = (-5)^x\). There was already a difference between these two. She wrote -1 in two tables. She asked the value of \( 5^{-1} \), and then looked at the students. They did not say anything. The students were talking to each other, but they were not talking about the question.

Sara wrote \( n^{-x} = \frac{1}{n^x} \). Then, she applied it for the equations. She wrote \( 5^{-1} = \frac{1}{5^1} \) for the equation \((y = -(5^x))\). Then one of the female students told Sara it would be \(-\frac{1}{5}\). Another female student asked if the \( y \) value would be positive for the second equation \((y = (-5)^x)\). Sara looked at the equation for a few seconds. She told the students it would be 1. Then, she decided to write it on the board. Sara wrote \((-5)^{-1} = \frac{1}{-5^1}\). She said it would be \(\frac{1}{-5}\) and wrote it in the table. One of the female students told Sara that she did not understand the difference between these two equations. Sara told the student that only one thing was different. Sara pointed out with her finger (on the board when \( x \) was zero its corresponding value \( y \) was different in two equations.) She told the student that they would try for a couple more points, that the graph of these equations would show the big difference between them. She told them “let’s try for 2.” She
asked them what y value would be for \( y = -5^x \). One of the students answered \(-5^2\). Sara wrote the student’s answer in the table. Sara asked the same question for \( y = (-5)^x \). The students were quiet. She waited for a few seconds, and wrote 25 in the table. Sara wrote 3 for x value in the two tables. She asked them when x was 3, what y value would be. The male student, Mike answered the question. He said it was -125 for \( y = -\left(5^x\right) \). Sara wrote his answer in the table.

One of the female students said it was 125 for \( y = (-5)^3 \). Sara asked if it was positive or negative. She also asked the student if she was sure. The student told Sara it would be -125. Sara asked her why she thought it was negative. The student explained that -5 times -5 would be 25. 25 times -5 would be -125. Sara wrote -125 in the table for \( y = (-5)^3 \). She also asked the students if they saw the pattern in this table. The male student, Mike, said every other y value was negative in \( y = (-5)^x \). Sara told him that he was correct.

Sara asked students to graph these equations using the table values that were on the board. One of the students asked Sara if she could stop graphing at -125. Sara told the student that she could stop at the value of -125. However, Sara told her to estimate where -125 would be on the graph. She let the students work on graphing for a couple of minutes. One of the female students asked Sara why she chose negative numbers on both of the tables. Sara said, “I was just throwing numbers, we could have used anything. It doesn’t matter what we use. We are just trying to find some points.” The student did not say anything after this explanation. The students continued to work on graphing the equations. In the meantime, Sara erased everything on the board, and then wrote the equations \( y = -\left(5^x\right), y = (-5)^x \). Under these equations, she drew two Cartesian coordinates. She asked the students to tell her x and y values for \( y = -\left(5^x\right) \). She plotted the points. Sara told the class this was how the graph looked. One of the female students asked what was the use of the graph. Sara told the student, “I am going to get to that later.” She moved onto \( y = (-5)^3 \). Like the previous graph, she asked students to tell her the points. Then, Sara plotted the points using dots. She connected the points. Sara said, “This one looked like going to negative. Do you see the difference?” The students were quite and did not answer to her question. On the graph, she extended the graph in the second quadrant. Then, she
told the students, “It would go like this.” The following excerpt indicates how Sara explained these two graphs \( y = -5^x \) and \( y = (-5)^x \) were different;

The student: Why graph looks different?
Sara: Well, it just because like [Pause] you mean why it’s going straight down.”
The student: Why it is not going down?
Sara: We can plug in more points and checked, but this is what it does. This one is an isolated function. It goes up and down like that. It keeps getting bigger and bigger.
The student: Did you say isolated function?
Sara: Isolated.
The student: What is that mean?
Sara: It goes up and down.
The student: What is the other one called?
Sara: This one flipped from the normal function because remember the other function looked like this [First example]
The student: That’s \( f(x) = 4^x \) regular function. That one \( y = -5^x \) is flipped function?
Sara: Yes, when there is a negative on the outside. It makes all the values negative.
The student: And the other one?
Sara: Isolated. What I wanted to see that for any of these exponential functions. Really, any functions that are given, in order to graph it, all you have to do is just make up a little chart [table], x and y. Plugged in points, find points in. You can eventually just figure out how the graph goes. Always a good trick, but exponential functions, this one is special case [graph of \( y = -5^x \)] but otherwise they always go really up. They go for a long time, they just close to zero. All of a sudden they go really fast. Go down really fast, but that’s always how the exponential function is. You [the student] asked why we have these. What is the point? They are very useful. They model things in the physical world. They represent things either really grow quickly or decrease really quickly.

Sara asked the students if they could think of any example. A few students gave examples of exponential functions such as sale taxes, death rate, stocks and food prices. After the students’ examples, Sara gave an example for a person’s height rate. She told the students, “A person’s height will start slowly and then it will shoot up really fast.” Sara also said that in real life, it would not happen this way. In real life it would not go to infinity. After talking about real life examples of exponential functions, Sara decided to give another example. She stood in front of the students. First she told the students, many of them knew that “bunny rabbits” produce very quickly and very often. She found out that rabbits breed pretty much every month. As she was talking about rabbits, she often looked at her notes. Sara said,” Let’s make a graph.” She changed her mind. She decided to make a chart that had x and y values. “x” value represented months. She asked the students what “y” was. One of the students said that “y”
would present amount. Sara wrote “# rabbits” for y value. Sara told the students that they had 4 rabbits including 2 girls and 2 boys. Months would be zero at the beginning (see Figure 4.72).

<table>
<thead>
<tr>
<th>Months</th>
<th># Rabbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>y</td>
</tr>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Figure 4.72: Table values for rabbit example

After a month, Sara asked the students, “The two female rabbits will have two sets of babies right?” The students did not say anything. Then, she told them that there were two female rabbits. Each would have two babies. There would be four babies and four parents. She asked them, “There would be 8 rabbits, right?” The students did not say anything. She moved on and wrote 2 in x value.

Sara told the students that in real life it would take six month for rabbits to breed. In this example, she assumed it would take a month. Then, Sara decided to draw a diagram. She started from the beginning of the example. She drew a diagram for the rabbits. (see Figure 4.73). She started explaining the “bunny rabbit” activity from the beginning. They had 4 rabbits that were 2 girls and 2 boys. After a moth, a boy and girl would have two babies. Each month a boy and girl would have two babies as well. Then, she said, “This isn’t easy.” Sara erased her diagram and table.

She asked the students if they could see that rabbits would produce every month. They would have more and more babies. It would keep growing every month. Sara drew a Cartesian coordinate system on the board. She drew the number of rabbits versus month graph without using any points. She said, “It will just shoots up. This is the point of having exponential functions. These are the things in the real world and you need to predict.” Then, she finished the lesson 8 minutes early.
Analysis of Sara’s teaching

Sara asked students to graph the equation \( f(x) = 4^x \). She told students to use any value for \( x \) and \( y \). Sara thought any point could be used for graphing. She suggested that finding the table values by substituting any numbers. She failed to see that the graph would not be informative. In other words, the graph would be very close to origin or the point would be very far from each other. For instance, one of the students told Sara that she had an odd graph due the \( x \) and \( y \) values.

Sara thought that \( y = (-5)^x \) was an exponential function. This showed that Sara did not know the how the exponential function was defined. She did not see that exponential function, \( f \) given by \( f(x) = a^x \) for every \( x \) in \( R \), where \( a > 0 \) and \( a \neq 0 \). When Sara drew the graph of \( y = (-5)^x \), she did not recognize the graph did not represent the behavior of exponential function. In other words, Sara did not see the relation that was not an exponential function from the equation or the graph.

At last, Sara gave the “bunny rabbit” example. In this example, the rabbits give birth after one month of pregnancy. At the end of their second month, the rabbits breed again. Sara’s goal was to have a discussion about how the exponential functions would be a model for growth of the rabbits. She wanted to show the students how the exponential functions were used and represented in real life. However, she failed to represent the growth of the rabbits using table or diagram. She did not think of asking the students to find the relationship between the number of rabbits and rate. She could have composed a function that would represent rabbits at any time \( (y = ab^x, \ y = (\# \text{ rabbits})(\text{rate}^x)) \). Throughout the lesson, Sara did not give enough time to the students to think about her questions. She asked questions, but she answered most of her questions. When students asked questions of her, Sara often could not give them an explanation.

Sara was asked to evaluate and reflect on her teaching of exponential functions to four female and one male high school seniors at the Southern High School. She was asked to explain how she thought the lesson went in the class. She thought that the students did well, but she expected them to know more about exponential functions. She said, “They were all pretty good. Only thing, they didn’t know as much as they needed to know exponential functions. That was an issue.”
Sara found it was difficult to teach the students that did not understand very easily. She asked students to substitute any number for x values and find the y values. She asked them to try graphing the equation on their worksheet. Sara did not recognize that substituting any number was not appropriate for graphing the equation. She thought if the students substituted different numbers, they could draw the graph of equation.

It’s difficult when the student doesn’t get it. When I was going around and checking their work or actually as soon as I gave them little sheets of paper they’re all like okay I want all to try and graph these functions. When they were working on it, I was looking at their paper they would do like two points and try because I guess they’re used to graphing lines. You just graph, you just plot two points and then connect the dots and draw lines. I was like try another one. Do a couple points to figure out what the graph looks like. That can be difficult when they’re not getting it. They’re not doing what you want them to do. Expected them to try a bunch of points, they were just sitting there looking up to thinking that they had to graph from that.

When I asked her why she thought the students did not know, Sara referred to negative exponents in her explanation. Then, Sara explained that the students did not know how to find the value of negative exponents. However, she said, “Well, they didn’t understand that like negative exponents like $4^{-1} = \frac{1}{4}$. They didn’t know that. You need to be able to how to find. You need to know to do exponential function.”

When Sara wrote her lesson plan, she wanted to use “money example” for the students. In her teaching, Sara gave worksheet to each student. The worksheet had one of the five different functions, $y = 3^x, \; y = 4^x, \; y = 5^x, \; y = -5^x$ and $y = (-5)^x$. Each student received one of these five functions.

Well, I gave them each little sheet of paper. They had just like a very strait forward exponential function like $y = 3^x, \; y = 4^x, \; y = 5^x$ and each got one of those and then on each sheet of paper there was another one that was different like $y = (-5)^x$ and $y = -5^x$ something like that. I wanted to have them all come up to the board and have each graph up there or have them graph what they came up with and then discussed it, but there wasn’t room for that and they weren’t getting it. They didn’t because they couldn’t do negative powers. They didn’t know how to do that. All their graphs, they would draw like a three points. That would be like a straight line. Because they didn’t weren’t able to get the numbers right. So, I walked around and I saw the graphs that they were coming up with weren’t. None of them were right. They better just do one with that group on the board, and then she [student] asked about these I thought I show them the other example since she was curious.
As it can be seen from the excerpt above, Sara asked students to graph the equation that was on their worksheet. When she checked students’ work, Sara realized the students could not graph it because they could not find the values for x and y correctly.

Sara was asked whether or not the students were able to understand what an exponential function was and how it was used at the end of the lesson.

We had discussed, they all came up with things that could be used to represent. So, I think they did understand that. But the whole thing, it was difficult to [Pause] because they didn’t understand the negative exponents. It made it hard to understand graphing part of it. I think they understood what it could be used for.

As it can be seen from the excerpt above, Sara recognized that the students did not understand negative exponents. They had difficulty with graphing the equation. In her teaching, most of the students did not know both zero and negative exponents. Even though Sara saw that the students did not know the exponential notations, she wrote the definition \((a^{-n} = \frac{1}{a^n})\) and illustrated numerical examples \((4^{-1} = \frac{1}{4})\). However, she did not give any explanation for the exponential notations or ask the students whether or not they understood. She did not write on the board the definition of zero exponents. She asked them verbally. For instance, Sara verbally asked when x was zero, what the value of \(4^0\) would be. She answered her question and told the students \(4^0\) would be equal to 1. When one of the students asked why \(4^0\) was equal to 1, she could not explain why it was equal to 1.

I showed her a video clip from her teaching, indicating that she drew the graph of two equations \((y = -5^x), y = (-5)^x\) and then gave a “bunny rabbit” example.

In her lesson, Sara planned to use the “money example.” During her teaching, Sara decided not to use it for the lesson. Instead of the “money example”, she used “bunny rabbit” examples instead of “money example.” I asked Sara why she changed the example.

I was going to do one example at the end with the money thing. I didn’t feel like that would work out because we had already gone over it. I tried to do a different example but that didn’t really work out either. I think they got the point.

I asked Sara to explain why she chose these two equations, \(y = -5^x\) and \(y = (-5)^x\).

She selected these examples because Sara wanted to give different examples to the students.

I just wanted to try out, give them all different exponential functions to try. I thought about not doing \(y = (-5)^x\), just because it was the isolating function. It’s confusing and it doesn’t, it’s just not to typical exponential function, but I was like well I’ll let them see it
anyway, but I wasn’t able to explain it very well though. I remember learning about it. I just [Pause] she [one of the female students] asked why does it do that? I was like, well, it just does. I just wasn’t able to explain that very well. They also asked some other things that I wasn’t sure what to say like number to the zero power equals one. They asked why. I was like, well, no one ever taught me why. I don’t know really.

As it can be seen from the excerpt above, Sara did not recognize that \( y = (-5)^x \) was not an exponential function. She did not remember that exponential function \( f \) is given by \( f(x) = a^x \) for every \( x \) in \( R \), where \( a > 0 \) or \( 0 < a < 1 \), and \( a \neq 0 \).

Sara included “bunny rabbit” activity after she was interviewed about her lesson plan. She decided to include this example because Sara wanted data that could be seen in real life. She thought “bunny rabbit” activity was better than the “money example” that was in her lesson plan.

I wanted to try to have like graphs, the different real data on it. So, I didn’t find a graph of that but I did try the information looks like. So I’ll just go through. We’ll figure it out together. It was, I sort of have that on the lesson plan but not really because I kind of thought of it later. So, I didn’t have it on the lesson plan but I wanted to do it. I just thought that was a good. Like [Pause] a sort of commonly used example for exponential functions, rabbit reproduction. I was like, well, I just find out how often they reproduce. I do a little graph of that.

When Sara gave the “bunny rabbit” activity in the class, she thought the activity was not interesting. She could not explain the example to the students very well. She said, “I thought that later on I was thinking about it how I could have better explained it because I just wasn’t doing a good job explaining it and it wasn’t very interesting.” Sara could not explain or draw the graph of number of rabbits versus months, when she utilized this example. She tried to explain that the relationship between the number of rabbits and months was an exponential function without plotting the values.

In her teaching, Sara began to talk about the “bunny rabbit”; then she gave a couple of examples and wrote the data on the table. Then, Sara erased everything on the board. I asked Sara why she decided not to find the number of rabbits.

I had done the example in the morning and I was trying to sort of going through it quickly. I did it in the morning before I went to class and yes, I was trying to go quickly through. I was trying to explain all the stuff, and just sort of I kept saying this sort of extreme information that I didn’t need to say. I just got confused since it’s exponential. I kept track of the numbers in my head. I just got confused and I felt like I was sort of
losing their attention. So, I just skipped ahead and okay here is the point. This isn’t really going too well. I should have thought it through more.

As can be seen from the excerpt above, Sara worked on the “bunny rabbit” activity in the morning before she went to class to teach her lesson. As she was explaining, she got confused. She gave a brief explanation about the example.

I asked Sara how the students were involved in class activities. She thought the five students paid attention to the class activities.

So, they were all participating in it, calling out the answers and asking questions. When they didn’t understand the negative power, I showed them. After a while, there was just a one girl who was talking a lot, but they were all paying attention. I think they participated well.

She graphed and answered most of the questions. When Sara asked questions of the students, she did not give enough time for them to think about the answers.

Sara was asked what she thought the students learned about exponential functions at the end of this lesson. Sara said she believed the students understood the graph of the exponential function

Well, I think they understood what they could represent. What the graph looks like even though I may have confused them with isolating function; that may have thrown them off little bit, but I think they understood what the graph of an exponential looks like.

When I asked Sara whether or not she achieved her goals for this lesson, she said she thought the students needed more practice to graph exponential functions. She believed that the students understood the basic concept in exponential function and identified data that displayed exponential behavior.

Pretty well, but they could use a lot more practice to really be able to do this. I know they weren’t very comfortable with the negative powers. They definitely needed more practice, but I think they always got a basis to be able to do these things. I did feel like they were good identifying data at this place, exponential behavior.

Even though Sara thought she achieved the goals for this lesson, the students were not able to graph the equation. Most of them did not know negative and zero exponents. Sara used the “bunny rabbit” activity to show data that would represent exponential behavior. She could not show the exponential behavior in the “bunny rabbit” activity because she could not represent the data. She could not utilize table values to show that the relationship between the number of rabbits and months was an exponential function.
Sara was asked what she would write for a reflection of this lesson. Overall, Sara thought the students paid attention to the class activities. The lesson went well. However, she felt it would have been better if she had focused on the negative exponents in this lesson.

I felt like they paid attention. It went pretty well. They just kind of threw me off because they didn’t understand negative powers so that I probably would have done, it probably would have been appropriate for me to teach them about that instead of exponential function. [Pause] but they seemed to at least understand what it was used for, and they paid attention. It went pretty well, though, I think.

I also asked Sara If she would change anything on her teaching of the lesson and why she would change it. She said she wanted to change the examples she used in this lesson. She thought basic examples would be better then different exponential functions. She felt that she would explain the “bunny rabbits” activity better.

Well, the example I used, I would probably. I would want that to go smooth. I would want either to do a better job of explaining that example or use a different example that was more interesting. I think if I had done a better job of explaining it, then it would have been more interesting. I would have kept their attention with that. It does turn out to be a nice example, if you do it properly. So, I’d change that maybe I would not give them as many types of functions may be give them the regular function first and then discuss a few different other kinds. Maybe it would be just move function to the left or to the right or either up or down or flip it. Maybe, use those three different types of transitions.

As it can be seen from the excerpt above, Sara recognized that she should choose basic examples of exponential functions and give a better explanation to the students. However, she did not mention what these examples would be. She did not recognize that a class discussion is as important as explanations. It is also important to have students share and represent their findings.

I asked Sara how she would explain the example better. Sara noticed that she gave unnecessary information for the “bunny rabbit” activity.

I feel like when I was trying to explain that example, I just got little frustrated. I didn’t feel like I was doing a very good job of explaining it. I could have been more conscience with what I said because I went on about how I would through a little side note. Well usually rabbits don’t start reproducing until they are 5 moths old. For this example, we just assumed that we just do it this way and I just kept throwing these little things that I didn’t need to say. This may have confused the students. It just wasn’t necessary for me to say that so I would prefer if I hadn’t just throw these little details in.
As can be seen from the excerpt above, Sara did not see that it was important to show the students how each month the number of rabbits increased. She could have shown the number of rabbits in the first couple of months. Then Sara could have asked the students to find the data and graph it in groups.

I asked Sara if she thought the students might have problems with negative exponents before she thought out the lesson.

I think. Well, I don’t think that I considered it. They wouldn’t understand how to do that. I even ran through the lesson with one of my roommates. Trying to make sure that it flowed well. I didn’t run through that rabbit example with her. Yeah, I tried to think of things that they would say or problems they might have, but I just didn’t know or I never thought anything before any way. So it was just kind of a first time for me.

Due to her lack of teaching experience, Sara did not think of the students’ difficulties. Even though she noticed their difficulties with negative and zero exponents, she just gave them the rule. She had practiced the teaching of her lesson with a roommate, but she did not recognize that the equation, $y = (-5)^x$, was not an exponential function.

I asked Sara whether she considered whether the students might have difficulty learning exponential functions prior to the teaching of her lesson. She did not think they could not obtain the value of negative exponents.

I just didn’t have any experience with what students may or may not know so this would give me a better idea in future to expect. Maybe, or at least realize maybe students won’t even have enough foundation in math then understand how to do what I am trying to teach them. So, I didn’t. I mean I didn’t think about the fact that they couldn’t do the negative exponents. I just assumed that they would be able to since I’m teaching them about exponential functions. I figured they must have already known the stuff they needed to do exponential functions. So, I didn’t really think of that. Yeah, I knew there was a possibility that they could be confused about anything.

**Researcher’s observation about analysis of teaching of the lesson**

Sara did not mention how to draw $f(x) = 4^x$. She asked the students to try any values for $x$ and find the values of $y$. Sara’s idea of making of a table and substituting any values for $x$ to draw the graph of $f(x) = 4^x$ was not appropriate since the students were not did not learn the definitions and properties of exponential functions. However, when she drew the graph, Sara did not mention any properties of this function (increasing, decreasing, domain, range, horizontal asymptote, symmetry). I was hoping that at this point she would go back and discuss the
definition of exponential function and its properties. This clearly shows that her limited subject matter knowledge made it more difficult for her to ask appropriate initial questions or offer explanations that made the concept easier for the students. The impact of her limited or weak subject matter knowledge and pedagogical content knowledge on her instruction was obvious in her teaching. Again, Sara created table values for two equations \( y = -5^x \), \( y = (-5)^x \). She utilized the point-wise approach to for table values. Then, she drew the graphs without discussing the behavior of the function. She asked the students what the difference between the graphs was, but she moved on to the next example without discussing anything. When the student did not see the difference between these two, she was unable to explain why these graphs were different. In addition, even after she drew the graph of \( y = (-5)^x \), she failed to recognize that this was not an exponential function. Up to this point, her teaching was teacher-centered because she could not ask appropriate questions to get students involved or discuss the exponential functions.

At this point, I was hoping that she would give an example of exponential function whose base was between 0 and 1. However, Sara preferred to discuss the activity (bunny rabbit). She created a chart for x and y values. “x” value represented months. She asked the students what “y” was. One of the students said that “y” would present an amount. Sara wrote “# rabbits” for y value. Sara told the students that they had 4 rabbits which included 2 girls and 2 boys. Months would be zero at the beginning. As she was explaining, Sara got confused and then created a diagram. She decided to erase everything on the board. She drew the graph representing the relationship between the number of rabbits and months without using x and y values. After drawing the graph, she asked the students if they saw how the number of rabbits increased every month. She said, “it [the graph] will just shoots up.”

During our interview after her teaching, Sara watched the video clip and told me that she could not explain the activity. She could not communicate with students or get them engaged in the class. Even though the students did not understand how to graph exponential functions, she continued the lesson because she thought it was necessary to finish the lesson as planned. She mentioned students’ difficulties when she was calculating the negative exponents of numbers \( (4^{-1}, 4^{-2}) \). She believed that she reached the goals of her lesson plan. When I asked her what she would change if she taught this class again, Sara said she would explain the examples and activity better. She would choose a more interesting activity for the lesson. She said, she would
select basic functions. So, she would move function to the left or to the right or either up or down or flip it.

Sara did not have any teaching experience until she taught this class. Basically, this was her first time teaching a class. I believe that this was one of the biggest factors that prevented her from communicating with students or getting students engaged in the class. She is a novice teacher and she will learn and improve her teaching skills with experience. However, during the evaluation and reflection of her teaching, Sara did not talk about the organization of her lesson plan (introduction, examples, activities, etc.). Throughout the class, she drew the graphs by plotting points (point-wise approach) and did not mention the relationships between her examples \( y = 2^x, y = -5^x \). She did not even define the behavior of the exponential function. Even though she noticed that the students did not understand the graphing exponential functions. She did not know why they did not understand. Her approach to the graph \( f(x) = 4^x \) was not appropriate but she did not discuss the behavior of the function for very large or small \( x \) values. If the students knew the definition of an exponential function and its properties, it would have been easy for Sara to explain the concept and engage the students in the class.

Sara did not have any experience in teaching. Basically, this was her first teaching experience. I believe that one of the biggest factors that prevented her from communicating with students or getting students engaged in the class was her inexperience. Sara is a novice teacher, and she will learn and improve her teaching skills by teaching. The improvement process requires novice teachers to recognize and build upon their experiences, and reflects on those experiences (Richardson, 1990). When they interact with their existing understanding and experiences, new understanding occurs (Richardson, 1994).
CHAPTER 5

CONCLUSION

This study that was conducted with the two prospective secondary teachers from a state university in southern Florida was an exploratory study designed to examine these prospective secondary teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function as they participated in tasks during the six weeks of data collection. These tasks included depicting mathematics problems, and addressing different aspects of the concept of function, as well as organizing the use of different representations. In this study, the prospective secondary teachers’ subject matter knowledge and pedagogical content knowledge was analyzed and assessed using the framework of Even (1989). The model of Wilson, Shulman and Richert (1987) was used to organize the tasks and examined whether or not they improved their pedagogical content knowledge as a result of doing these tasks. The purpose of this chapter is mainly to discuss and summarize the findings unraveled during the analysis, and to outline the implications of the study.

The summary deriving from the study is discussed in relation to the three research questions by: (1) the nature of prospective secondary mathematics teachers’ subject matter knowledge and how they classify relations into functions and non-functions; (2) prospective secondary mathematics teachers’ understanding of the relationships among different types of functions and how they transfer from one representation to another; and (3) the nature of prospective secondary mathematics teachers’ pedagogical content knowledge for teaching the concept of function, and the nature of the relationships between prospective secondary mathematics teachers’ subject matter knowledge and pedagogical content knowledge for teaching the concept of function.

Research question one the first question was regarding the nature of prospective secondary mathematics teachers’ subject matter knowledge for teaching the concept of function and how they classified relations into functions and non-functions. In the function questionnaire, the participants mentioned the nature of univalence property and used the vertical line test
excessively to determine whether a relation was a function or not. The participants were trying to transform the equation (or a table) into graphical form so that they could use the vertical line test. However, they did not explain why the vertical line test worked or what it meant to fail the test. The participants also did not know the arbitrary nature of functions. They were thinking that all functions can be described by an equation or that functions are subsets of equations. According to them, functions were smooth and continuous most of the time. In the function questionnaire, the participants recognized different representation of functions and non-functions (equation, graph, set notations). However, they had the tendency to use the vertical line test instead of using other strategies (equation). The participants were unable to make a connection between graph and equation. They could not determine whether or not the equation was a function. Therefore, the participants tried to find out what the graph of the equation looked like and then utilized the vertical line test to determine whether the relation was a function or not. The result of the study revealed that participants did not refer to the arbitrary nature of functions. It also revealed the excessive use of the vertical line test without explaining what it meant to fail the test, why it worked, or why it was or was not a function.

Research question two looked at the prospective secondary mathematics teachers’ understanding of the relationships among different types of functions and how they transferred from one representation to another. In the card-sorting activity, the participants were able to sort the functions according to their representations. It was easier for them to recognize functions represented as a graph and/or equation. However, they had difficulty determining functions given verbally (i.e., word problems) or in a table. Especially when the function was given in the table, the participants tended to use a point-wise approach to match the points to a graph of familiar functions. Overall, they had difficulty translating from verbal and table representations. Their understanding of functions among different representations was very weak and lacked connections. Jack wrote an exponential equation to represent the logarithmic function. In other words, he recognized the graph of logarithmic function but could not write a correct equation. This indicates that his graph and algebraic representations of logarithmic and exponential functions were not connected and very weak. Similar difficulties surfaced when participants dealt with different representations of quadratic functions. They were able to recognize the different representations of quadratic functions in card sorting. The result of their responses revealed that the participants’ subject matter knowledge was weak and superficial.
The third question I intended to answer was the nature of the prospective secondary mathematics teachers’ pedagogical content knowledge for teaching the concept of function and the nature of relationships between their subject matter knowledge and pedagogical content knowledge for teaching the concept of function.

When the participants were analyzing the student’s incorrect estimation, their approach to the problems was different. Jack’s approach was student-centered. He wanted to engage the student. This clearly indicated that he had a good idea of how to help the students but failed to come up with a specific solution strategy due to his limited pedagogical content knowledge. On the other hand, Sara’s approach was teacher-centered. She wanted to solve the problems instead of allowing the student to do so. If the participants had had strong pedagogical content knowledge or had known more about common misconceptions about student’s mistakes, they might have suggested a more appropriate strategy to help the student. Their responses indicated their difficulties recognizing the sources of students’ mistakes as well as limited pedagogical content knowledge about students’ misconceptions of functions. It became clear that the participants were not taking the student’s possible difficulties or misconceptions about functions into account. For instance, giving an example of non-function can be very helpful. A teacher should know or anticipate sources of students’ common mistakes and take them into account when making instructional decisions. Otherwise, if the teacher gives typical examples of functions, students may have a different understanding of the concept of function, as explained in the analysis of students’ incorrect solutions, the participants utilized their pedagogical content knowledge to help the students overcome the problem (misconception).

Since pedagogical content knowledge was weak and subject matter knowledge was superficial, they could not take the sources of the students’ incorrect answers and possible misconceptions about functions into account when they were trying to help the student.

Because of their weak subject matter knowledge, the participants recognized some typical graphs and equations but had difficulties recognizing functions in different representations. This difficulty was central to manner in which each prospective teacher handled student’s errors. However, they did not take the misconceptions the student had about functions into account. The participants did not think of using constant functions or piece-wise functions to help the student create correct image. For instance, on the third analysis of the question on the definition of a function, Jack was thinking of using the vertical line test as an example of what the function
was, overall, before preparing a lesson plan and teaching a lesson plan. By looking at the analysis of student’s incorrect solutions, it was observed that while the participants utilized their pedagogical content knowledge to find or suggest an appropriate way to help the student, weak subject matter knowledge as well as weak pedagogical content knowledge prevented them from taking student’s misconceptions and sources of incorrect solutions into account. When the participants were describing the strategy to make it easier for the students, during their lesson analysis and teaching, their weak subject matter knowledge led them to select an inappropriate activity and examples of exponential functions. Because of weak subject matter knowledge they were unable to chose appropriate questions or organize a lesson plan. Because of their weak pedagogical content knowledge, the participants failed to ask the appropriate questions to make the concept easier for students. Most of the time, they were unaware of students thinking and understanding. As a result, they ended up dominating the class, asking and answering questions without the help of the students.

The participant, Jack, had taken pedagogical classes and utilized both his subject matter knowledge and pedagogical content knowledge when he analyzed student’s incorrect solutions. Unlike him, Sara had not completed any pedagogical classes at the time of the study. Both of them had very different approaches to student’s incorrect solutions. For example, Jack wanted to have students measure x and y to overcome an estimation mistake and asking the same questions to all students and having them share their class with classmates. Sara wanted to measure x and y to overcome an estimation mistake. However, due to their limited and weak subject matter knowledge and pedagogical content knowledge, Jack and Sara could not choose a specific strategy. These participants could not take the sources of students’ incorrect solutions or misconceptions about students’ functions (smooth, line, continuous) into account.

Similar problems surfaced during their teaching of exponential functions, even though the goal of the lesson was to be able to graph exponential functions and display exponential behavior using data. The participants failed to make appropriate instructional decisions to make graphing easier for the students. For example, not defining exponential functions, horizontal shift, vertical shift, and their relations with the original functions. After following the model of Wilson, Shulman and Richert (1987), the participants did gain some experience in preparing and teaching a lesson; however, their instructional strategy for teaching exponential functions changed very little due to their limited subject matter knowledge and pedagogical content knowledge. The
participants did not know the criteria for graphing exponential functions. They did not know how to teach graphing exponential functions. They failed to challenge the students and ask critical questions. They could not choose appropriate exponential functions due to deficiencies in their subject matter knowledge. They also could not determine the necessary components in exponential functions that students needed to know and to draw the graph of exponential functions. In addition, the participants had trouble explaining to the students because their pedagogical content knowledge for teaching was not comprehensive and articulated. However, during evaluation and reflection, they noticed the students’ difficulties and lack of understanding and offered some instructional strategies to help them.

The result of this study is also consistent with the study of Even (1989). Even concluded from her findings that prospective teachers’ knowledge of functions was weak and fragile and they had difficulties working with different representations of functions (Even, 1990, 1998). She indicated that a teacher’s subject matter knowledge influences his/her way of teaching the students. The teacher who has strong mathematical knowledge is more competent in helping his/her students attain a meaningful understanding of the subject matter. The teacher asks questions, stimulates discussions, and suggests different points of view to the students. These activities and decisions require teachers to have adequate subject matter knowledge as well as pedagogical content knowledge (Even, 1989).

The result of the study is consistent with the study of Sanchez and Llinares (2003). They concluded that many of the prospective teachers had trouble communicating because their pedagogical content knowledge for teaching was not comprehensive and articulated. When prospective teachers have misconceptions or limited content knowledge, they may pass on these misconceptions to their students, fail to challenge them (Ball & McDiarmid, 1990). Prospective teachers’ conceptions might limit their ability to present subject matter in appropriate ways, give helpful explanations and conduct discussions (Even & Tirosh, 1995).

The result of this study is consistent with Ball (1990c), who stated that subject matter knowledge should be a central focus of teacher education in order to teach mathematics effectively. I believe that both subject matter knowledge and pedagogical content knowledge should be a central focus in teacher education. In order to teach effectively, teachers’ understanding of the concept of function must rely on solid knowledge of the subject matter. Moreover, under different circumstances, the teacher should be able to recognize the pedagogical
grounds for selecting alternative ways of representing the subject matter because the quality of teaching must rest on features of pedagogical reasoning that can be explained through pedagogical actions (Shulman, 2004). Prospective teachers develop their pedagogical content knowledge as they plan to teach as well as during actual teaching (Wilson, Shulman & Richert, 1987). In this study, even though the participants did not make big improvements, during the lesson plan analysis, teaching and the evaluation of their teaching, they pointed out important issues: strong and weak aspects of lesson plans, things to change in their lesson, weakness and strength of their teaching.

**Limitations of the Study**

There were a few limitations in this study. A larger sample size would have allowed the researcher to compare the results to the findings from the studies described in the literature (e.g., Even, 1989; Sanchez & Llinares, 2003) study. The researcher’s lack of experience in conducting qualitative research was also a limitation in this study. One of the characteristics of qualitative research is that the researcher is the primary instrument for data collection and analysis. This can result in certain limitations and biases that will affect the study. The participants’ lack of experience for preparing lesson plan and teaching the lesson was also a limitation in this study.

**Implications for Teaching and Future Research**

Functions are a central area in the mathematics curriculum. This topic is also rich in that all the process standards (NCTM, 2000) can be emphasized during meaningful instruction in this area. Thus, based on this study, there are two implications for mathematics education programs and mathematics teacher educators.

In teacher education, the concept of function should be given increased attention. The focus in mathematics department courses should be multiple representations of functions as well as among representations of these functions. In this study, I observed that the participants failed to make connections among different representations to solve the problems (Janvier, 1987; Lesh, 1987; Lesh, Behr, & Post, 1987; NCTM, 2000). I suggest instructors utilize activities, including functions and multiple representations, translate from one representation to another, and manipulate the functions within the representations.

Instructors of teacher education programs should introduce prospective secondary teachers to basic functions found in the high school curriculum. Prospective secondary teachers should be able to go beyond their typical graph and equation. They should be able to connect all
these representations in a meaningful manner. Prospective teachers should be able to move from one function to another (e.g., exponential and logarithmic) using different representations.

In addition, prospective secondary teachers should prepare lesson plans, video record their lessons, and evaluate and reflect on their teaching under the supervision of their instructors because “knowledge is developed through cycles of planning, implementing, and reflecting on lessons (Fernández, 2005, p. 38).” Therefore, the model of Wilson, Shulman and Richert (1987) should be used to look for whether or not the prospective secondary teachers’ pedagogical content knowledge improves as a result of doing these tasks. The improvement of the prospective secondary teachers brings a new understanding that has been enhanced with increased awareness of the purpose of the instruction, and the subject matter. Increased awareness brings an enriched understanding. It may grow slowly, but a single such experience may bring to significant improvement (Wilson, Shulman & Richert, 1987). I believe with the use of this model, the prospective secondary teachers will enhance their pedagogical content knowledge for the concept of function as well as for their own professional growth. This model will also help mathematics educators better understand ways of incorporating these experiences in the professional development of teachers.

This research does not, by any means, gives an exhaustive study on understanding of the nature of prospective secondary teachers’ subject matter knowledge and pedagogical content knowledge of the concept of function. This study is just one of the springboards to achieve this goal.
APPENDIX-A
FUNCTION QUESTIONNAIRE

Written Task

S.S. number________________

Part I: Demographic Information
Please answer the following questions to the best of your ability

1. What is your age?_______________

2. What is your gender (please circle one)?   Male   Female

3. Please check your racial background:
   ______American Indian
   ______Asian or Pasific Islander
   ______Black
   ______Hispanic
   ______White Caucasian/ Non-Hispanic
   ______Other

4. Describe briefly any teaching experience you have had.

   __________________________________________________________
   __________________________________________________________
   __________________________________________________________

5. List below the names of the mathematics courses you completed in high school.

   __________________________________________________________
   __________________________________________________________
   __________________________________________________________
   __________________________________________________________
   __________________________________________________________
   __________________________________________________________

6. What is your current academic status?
   ______sophomore
   ______junior
   ______senior
7. What is your present overall college GPA? ________

8. List below the college mathematics courses you have completed (course numbers and names).

_________________________________  _______________________
_________________________________  _______________________
_________________________________  _______________________
_________________________________  _______________________
_________________________________  _______________________
_________________________________  _______________________
_________________________________  _______________________

9. List below the mathematics courses you are currently taking (course numbers and names).

_________________________________  _______________________
_________________________________  _______________________
_________________________________  _______________________

10. List below the mathematics education courses you have completed (numbers and names).

_________________________________  _______________________
_________________________________  _______________________
_________________________________  _______________________

11. List below the mathematics education courses you are currently taking (numbers and names).

_________________________________  _______________________
_________________________________  _______________________

12. Circle the grade level(s) you would prefer to teach (you may circle more than one).

5  6  7  8  9  10  11  12  no preference

13. Indicate which subject(s) you would prefer to teach (you may mark more than one).

_____ middle school mathematics
_____ algebra 1
_____ algebra 2
_____ geometry
_____ pre-calculus mathematics
_____ consumer mathematics
Part II: Mathematics Problems
The following exercise is designed to identify how you think about mathematical functions. This is not a “test,” and there are no “right” or “wrong” answers. Think of this as a tool for demonstrating how you think. Answer the questions to the best of your ability and do not refer to any outside sources while you work. Once you begin the questionnaire, please continue working until you complete it. It should take you less than one hour to complete.

While answering each question please describe your thinking-explain how and why you made your decisions, what other ideas, examples or questions came to your mind, etc. Answer the questions in order given as much as possible, but if you skip a question, change a response, or return to a question after completing it, please indicate what prompted your decision to do so. Please do not erase your original response. Try to describe your thinking even if you are not sure about how to answer a given question. Use additional sheets if necessary.

1. Give an example of a mathematical function.
2. Give a non-example of a function (something that is not a function).
3. Identify whether or not each of the following represents a mathematical function.
   Remember to describe your thinking as you work, particularly describing why you made your decision.
   3. \[2x - 3y = 5\]
   4. \[x^2 - y = 5\]
   5. \[x^2 + y^2 = 4\]
6. \{(1,2),(2,3),(3,4),(4,5)\}

7. \{(1,2)(-2,4)(3,4)(-1,1)\}

8. 

\[
\begin{align*}
\text{\textbf{y}} & \quad \text{\textbf{x}} \\
\end{align*}
\]
11. The relationship between the length of a side of a square and its area.

12. a) Give a definition of a function.

b) A student says that he/she does not understand this definition. Give an alternate version that might help the student understand.

13. If you substitute 1 for x in \( ax^2 + bx + c \) (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation \( ax^2 + bx + c = 0 \) have? Explain.
14. A student is asked to give an example of a graph of a function that passes through the points A and B (See fig. 1)

The student gives the following answer (See fig.2). When asked if there is another answer the student says: “No.” If you think the student is right- explain why. If you think the student is wrong-how many functions which satisfy the condition can you find? Explain.
16. This is the graph of the function \( f(x) = ax^2 + bx + c \).

State whether \( a, b \) and \( c \) are positive, negative or zero. Explain your decision.

17. A student was asked to find the equation of a line that goes through \( A \) and the origin \( O \).

She said: “Well I can use the line \( y=x \) as a reference line. The slope of line \( AO \) should be about twice the slope of the line \( y=x \), which is 1. So the slope of the line \( AO \) is about 2, and the equation is about \( y=2x \), let’s say \( y=1.9x \). What do you think the student had in mind? Is she right? Explain.
18. Explain to a student in algebra 2 how to sketch the graph of \( y = \frac{1}{x^2 - 1} \).

Can you illustrate what you expect the student to do? Are there other ways, which are more/less appropriate? Can you give examples? Why are they more/less appropriate?

19. A student said that there are 2 different inverse functions for the function \( f(x) = 10^x \). One is the root function and the other is the log function. Is the student right? Explain.
APPENDIX-B
CARD SORTING ACTIVITY

Functions can be represented in different ways including tables, graphs, equations, and verbal descriptions. In this section, examples of thirteen different functions will be presented representing five different categories of functions and four different representations for each category.

The following tasks are designed to focus your attention on different ways of categorizing functions and their representations. Each function has a number associated with it as indicated on the following pages where the functions are displayed. You will be asked to determine different ways of sorting the functions. What is important to consider is the criteria you used for the sorting task. Consider the following examples.

Example 1. Sort cards 1, 6, and 7 into piles based on the type of functions involved.

Solution. Since cards 6 and 7 represent quadratic equations and card 1 represents an exponential function, cards 6 and 7 should be classified together.

Example 2. Sort cards 1, 6, and 7 into piles based on the type of functions involved.

Solution. Since cards 6 and 7 represent quadratic equations and card 1 represents an exponential function, cards 6 and 7 should be classified together.

1. Consider cards 1, 12, and 27. What is the easiest way to sort these 3 cards into two piles? What criterion would be used?

2. If you were to sort 28 cards into four piles, which criterion makes the sorting easiest?

3. Sort cards 3, 15, and 19 into two piles. In what sense is your sorting criterion different from the criterion used in exercises 1 and 2?

4. In what sense are cards 6, 8, and 21 alike?
5. Sort cards 9, 13, 17, and 22 into different piles, two different ways. Place the cards into the following $2 \times 2$ matrix; $C_1$ and $C_2$ represent one type of criterion and $D_1$ and $D_2$ represent a different type of criterion.

\[
\begin{array}{cc}
C_1 & C_2 \\
D_1 &  \\
D_2 &  \\
\end{array}
\]

6. Sort cards 1, 6, 12, 16, 17, 21, 25, and 28 into two piles. Describe carefully the criterion you used to sort the cards.

7. Sort cards 1, 6, 7, 16, 18, 20, 21, 26, and 28 into three piles two different ways using two different criteria for sorting.
21. Workers use the following approach to locate the blockage of a sewage line. They start at the midpoint to determine which half of the line contains the blockage, thus decreasing the length of the line where the problem occurs by one-half. They then divide the remaining part of the line into two equal parts and repeat the process of determining which half of the line (in this section) contains the blockage. The partitioning process continues until the distance between the partition and the blockage is less than 8 feet. Mr. Davis wants to know the number of partitions necessary to guarantee finding the blockage within 8 feet for any line greater than 15 feet.

22. Central City has a population of 40,000 people. Recent studies indicate that over the past 20 years the population has steadily increased at an annual rate of 2%. Social scientists predict Central City will experience this same growth rate over the next 20 years.

23. The Art Museum has a bulletin showing its weekly hours: Monday: Closed Tuesday-Friday: 9 am-8 pm Saturday: 10 am-8 pm Sunday: Noon-8 pm

24. The number of hours the museum is open for each day in March.
APPENDIX-C
FIRST LESSON PLAN
THEORETICAL AND EXPERIMENTAL PROBABILITY

*Note: This lesson plan was created after the lesson was taught. No lesson plan was designed prior to the lesson—the class was taught from the book and without any preplanned examples.

I. Goals:
   • Students will be able to differentiate between experimental and theoretical probability.

II. Objectives:
   • Students will be able to define, compare, and contrast theoretical and experimental probability.
   • Students will be able to use theoretical and experimental probability to represent and solve real world problems.
   • Students will be able to compute probability for simple events

III. Sunshine State Standards addressed in the lesson:
   MA.E.2.4  The student identifies patterns and makes predictions from an orderly display of data using concepts of probability and statistics.
   MA.E.2.4.1 Determines probabilities using counting procedures, tables, tree diagrams, and formulas for permutations and combinations.
   MA.E.2.4.2 Determines the probability for simple and compound events as well as independent and dependent events.

IV. Materials needed
   Red and yellow integer chips

V. Grouping of students
   Students will work individually or will work in pairs throughout the lesson
VI. Time
   1 class period (50 minutes)

VII. Motivation
   A real world example of finding probability will be presented to the students (from the video… Rosa Parks)

VIII. Lesson procedure:
   1. Students will be given the definition of experimental probability.
   2. Students will be given the definition of theoretical probability.
   3. Students will be given examples of problems to find simple probabilities.
   4. Students will flip integer chips to compute and compare experimental and theoretical probabilities.

IX. Closure: Brings the whole lesson to a closure (not just ending class).

X. Extension: What other activities might be done if time permits?
   Students can roll two dice and compute the sum of the numbers 100 times and compare with an addition table to compute theoretical probabilities.

XI. Assessment: How will you know that the students have met the objectives and goals?
   Homework assigned form the lesson will be graded.

XII. Accommodations: Copies of notes handed out before or after lesson to students who need them. Students with physical disabilities can be paired with students who can do the experiment with them to help them out.

XIII. Reflections: on video
APPENDIX-D
SECOND LESSON PLAN
FUNDAMENTAL COUNTING PRINCIPLE

I. Goals:
Students will be able to understand and use the fundamental counting principle.

II. Objectives:
Students will discover the need to use the *Fundamental Counting Principles* and will logically state the formula for it.
Students will be able to use the fundamental counting principle to represent and solve real world problems.
Students will be able to compute the number of possible outcomes or options using the fundamental counting principle.

III. Sunshine State Standards addressed in the lesson:

MA.E.2.4 The student identifies patterns and makes predictions from an orderly display of data using concepts of probability and statistics.
MA.E.2.4.1 Determines probabilities using counting procedures, tables, tree diagrams, and formulas for permutations and combinations.

IV. Materials needed

V. Grouping of students
Students will work individually or will work in pairs throughout the lesson

VI. Time
1 class period (50 minutes)

VII. Motivation
Students will be presented with a real world problem and will be asked to work with their
partners to solve the problem. After pairs have worked the problems, the students will report their process for finding their responses as well as their responses. A discussion will ensue concerning the validity of the responses and the procedures for finding the correct answers.

VIII. Lesson procedure:
1. Following the discussion in the motivation section, students will be lead in to the direction of using some organized manner by which to display their data (with a tree). (Note—the teacher will never explicitly tell the students to use this, but will guide them to the use of it.)
2. Students will be presented with several real world problems in which the must use the principles of the fundamental counting principle to obtain correct answers. (see attached).
3. The class will come to the multiplication aspect of the principle through discussion of worked problems. Students will begin to see a need for (and want) a “shortcut” as stated by the Fundamental Counting Principle. Through questioning and discussion, this method of using multiplication of various choices will come out.
4. The teacher will formalize the language used by the students to present the standard way of relating the Fundamental Counting Principle.
5. A few more examples will be discussed and worked through, relating the tree diagram method, what the students intrinsically know to do, and the algorithm contained via the Fundamental Counting Principle.

IX. Closure
The students will write their own example of a real world problem involving the fundamental counting principle and share with class. Discussion will ensue as to the appropriate content and context of their problems in terms of it adhering to the use of the Fundamental Counting Principle.

X. Extension
Students can write another problem and trade with a partner to work their problem and discuss the appropriateness of each problem and the correctness of the answer.

XI. Assessment: How will you know that the students have met the objectives and goals?
Homework assigned form the lesson will be graded.

XII. Accommodations: Copies of notes handed out before or after lesson to students who
need them.

XIII. Reflections: on video

Daily Problem

Mr. A went to Publix to get a sub. He could choose white or wheat bread, turkey, ham, or meat beef meat, Swiss, American, or provolone cheese, and a small or large Coke. How many choices of meals can he have?

Tree Diagrams

- An organized way to count information
- Tree diagrams list all of the different possibilities, or outcomes, in a given situation

Probability

12.1 Counting Methods

The Stud

I’m Thirsty
Counting Principle

- To find the number of different outcomes for making choices in a sequence, multiply together the number of possibilities for each item.

I'm Thirsty

Start

Pepsi

Coke

Small Medium Large

Small Medium Large

Counting Principal

- To find the number of different outcomes for making choices in a sequence, multiply together the number of possibilities for each item.

Example 1

- A school has 3 math teachers, 4 English teachers, and 2 Spanish teachers. How many different sets of teachers could a student have for these 3 subjects?
Example 2

Earlene is dressing up for Halloween.
She can choose from 2 wigs (red or blond), 2 fake noses (bulbous or pointy),
and 2 pairs of glasses (green or mirrored). How many different disguises
can she create, and what are they?

Write your own!

Example 3

Coach Robinson has 5 pitchers, 2 catchers, and 2 shortstops on his team.
How many different sets of players can he use for these positions?
APPENDIX-E
LESSON PLAN GUIDELINE

**Goals:** General outcome(s) to be addressed through the lesson; broadly stated in terms of what each student is to achieve.

**Objectives:**
- The students will be able to graph exponential functions.
- The students will be able to identify data that displays exponential behavior.

**State Standards addressed in the lesson:**
- The student describes, analyzes, and generalizes relationships, patterns, and functions using words, symbols, variables, tables, and graphs.
- The student determines the impact when changing parameters of given functions.
- The student uses expressions, equations, graphs, and formulas to represent and interpret situations.

**Materials needed:**

**Grouping of students:**

**Time:**

**Motivation:** How will students initially be engaged in the lesson?

**Lesson procedure:** Set of instructions for the teacher; an outline of step-by-step description of what teacher and students will do during the lesson. Include questions to be asked, examples to be provided, problems students will work…Transition statements used to assist students to move from one activity to the next should be included.

**Closure:** Brings the whole lesson to a closure (not just ending class).

**Extension:** What other activities might be done if time permits.

**Assessment:** How will you know that the students have met the objectives and goals?

**Accommodations:** What modifications could be made to the lesson for students with learning disabilities or L.E.P. students?

**Reflections:** done after the lesson is taught. What went well? What needs be changed for the next time?
APPENDIX-F
JACK’S LESSON PLAN ON EXPONENTIAL FUNCTION

Goals: Students will be able identify examples of exponential functions – albeit represented through a problem, equation, graph, or data table. By the end of the lesson, students will have a general understanding of exponential functions’ rates and parameters.

Objectives:
- The student will be able to graph exponential functions.
- The student will be able to identify data that displays exponential behavior.

State Standards Addressed in Lesson:
- The student describes, analyzes, and generalizes relationships, patterns, and functions using words, symbols, variables, tables, and graphs.
- The student determines the impact when changing parameters of given functions.
- The student uses expressions, equations, graphs, and formulas to represent and interpret situations.

Materials needed: Ideally, the lesson requires a computer with a projector or simply an overhead projector. At the very least, the lesson plan requires a chalkboard. Students will need materials to write answers and draw graphs.

Grouping of students: When group discussion is assigned, students will work in groups of at least three and no more than four.

Time: Ideally, the lesson should fit within a 50 minute class period.

Motivation: Students will be drawn into the lesson initially through the ‘light versus water depth’ teaser problem.

Lesson procedure:
1. Assuming the students understand the basic concepts of linear equations and functions, the class will begin with the situation posed in the transparency. Students, in small groups, will discuss the situation and the possible answers to the three given questions. The students’ thoughts and answers will be revisited later in the lesson.
2. Students will be introduced to the idea of exponential functions, specifically starting with expressions that contain exponential expressions – those that have a variable as an exponent. [Example: \( y = 2^x \)]

   a. **Question**: Do you believe this exponential equation grows quicker than the previously studied \( y = x^2 \)?

      i. Students will form a hypothesis.
      ii. Individually, students will create a T-table with values ranging from at least -4 to 4 for both equations.
      iii. Students will graph the equations and formulate their conclusions.
      iv. **Question**: Do the two equations represent functions? Do the table(s)? Does the graph?
      v. **Question**: Do you recognize an asymptote anywhere on either of the two graphs? Can you think of an equation for the line of the asymptote?

b. On the same graph, students will then graph \( y = 4^x \) and will then be asked to explain the graphical effect of altering the exponential base from 2 to 4.

c. Students will then be asked to graph \( y = 3^x, y = 3^{x+2}, \) and \( y = 3^{x-2} \) on the same graph and will then be asked to explain the graphical effect of altering the exponent from \( x \) to \( x+2 \) and \( x-2 \). Students will then generalize the effect of \( k \) in an equation \( y = a^{x+k} \).

d. Students will then be asked to graph \( y = 2^x, y = 2^x + 2, \) and \( y = 2^x - 2 \) on the same graph and will then be asked to explain the graphical effect of altering the exponent from \( x \) to \( x+2 \) and \( x-2 \). Students will then generalize the effect of \( h \) in an equation \( y = a^x + h \).

e. Students will then graph \( y = 2^{x+1} + 3 \) and one student will present his or her findings to the class.

f. Students will then be asked to, as a group, reassess their thoughts and answers to the opening problem.

   i. **Question**: Could the relationship between the amount of light and level of water be related exponentially? Why or why not?
Closure: Students will make final conclusions on the opening problem based on their knowledge of exponential functions. Journals and/or exercises will be assigned for homework.

Extension: Possible extensions include real-world exponential problems (i.e.: interest rate) and instances where the base of the exponential term is a proper fraction.

Assessment: In the course of the period, several students will be called upon to volunteer their thoughts. Also, students will be given a homework assignment due at the next class period that will assess their ability to complete exercises geared at assessing the objectives of the day’s lesson. In addition, the student will turn in their written responses from their ‘light versus water depth’ group discussions. If a regular part of the class, journals may work tangent to (or in place of) their homework exercises.

Accommodations: Copies of the overhead could be distributed to those students with visual impairments. Groups could be divided in a certain fashion that would be advantageous to a certain student and their learning disability.

Reflections:

Transparency

Shedding Light on the Subject

Have you ever noticed how the amount of light differs the further you are under water? Consider the environment of the dolphins pictured and how the light intensity changes from the surface of the water to the bottom of the ocean.

1. How does the light change as the depth increases? Sketch a possible graph of the (depth, light intensity) relationship that you described.
2. If I(d) is the light intensity at a depth d, what does the quantity I(11) – I(10) represent?

3. How would the quantity I(12) – I(11) compare to I(11) – I(10)? What accounts for the difference?

Lesson Plan Outline

- Explain Goal: “My goal is to teach you how to graph exponential functions.”
- Activity: “To begin I am giving you each a function to graph. So work on these individually and if you have any questions let me know.”

Write each function on the board while waiting. Check to make sure the students are getting it. Have the students draw their graphs on the board. “ok I want you all to come up and draw your graphs on the board.”

Have students explain what they did. Choose someone who did it wrong first. Go through a couple that did it wrong before you get to the correct method. Try to get them figure it out on their own.

“So how did you get this graph?”

“Let’s see what someone else did.”

“How about you; how did you get this graph?”

Have students make commonalities.

“So what do you notice about these graphs?”

“How does it change when you add one?”

“What did you think will happen if you add 2?”

- Point out that when the number raised to the exponent is $> 0$, the function is increasing; when it is between 0 and 1, the function is decreasing, and when it’s between 0 and 1, the function is decreasing, and when it is $< 1$ it is oscillating.

Lesson Plan Outline

Transition: “Now, these are all nice graphs, but right now they don’t really mean anything, do they?”

Question: Does anyone have any idea what one of these graphs could be used for? What it could represent?

Explain that the x-axis represents time.

“What could the y-axis represent?”
“What stays steady for a long time and after a certain print increases rapidly?

Explain that experimental functions measure how quickly something grows or declines.

“What is an example of something that grows?”

Continue to discuss example of exponential functions.

I could get data representing the growth of a tree a person, a population, a stock, a business, temperature of a chemical, birth rate, climate change, half life of a substance, etc.

“Are these all exponential?”

“Are any exponential?”

They would probably behave exponential briefly, but not completely.

Explain that these functions are ideal; they are perfect representations of an imperfect model.

But they give us a good idea of what could happen.

Lesson Plan Outline

Transition: Money example

“Now I am going to do a little experiment.”

“You can probably gather from the fact that I want to be a teacher that…I don’t care that much about money!:)

“However, as we all know, it is a common goal in our American culture to want to get as much money as possible.”

“So what I want to know is this are you guys typical Americans…or are you like me?”

Put three exponential functions on the board.

\[ f(x) = 10^x, \quad f(x) = \left(\frac{4}{5}\right)^x, \quad f(x) = 2^x \]

- Ask, “who thinks they know what these functions will look like?”
- Have the students graph the functions.
- Have the students put the functions on the board.

“OK so if these x-axis measure time in months and the y-axis measures dollars, who wants #1, #2, #3?”

- “so now we know that this class is full of typical Americans :)” (most likely).

Lesson Plan Outline

Review:

“OK, so to review”
Question: what have been learning about today?

- Exponential functions
- Functions
- Increasing/decreasing graphs
- Growth/decay

Question: How to you plot the graph of an exponential function?

- Plug in numbers
- Make a table

Question: What are exponential functions used for? What do they represent?

- Stuff measured over time
- Stuff that increases/decreases all of a sudden population, birth rate, growth, decay, temperature, money, stocks, chemicals.

Very good 😊 Thanks you guys!
APPENDIX-H
INTERVIEW QUESTIONS

1. Would you please tell me your definition of a function?
2. Would you please give an example of a function?
3. Would you please give non-example of a function?
4. Can you give an equation describing your functions?
5. Does this graph have an equation?
6. What is the equation?
7. How do you know that is the right equation?
8. What are the coordinates of some points on the graph?
9. What does this equation mean to you?
10. Why did you connect the points with a line?
11. Could you graph this equation in any other method? Please explain.
12. Is it important to teach the line test for graphs of functions to students? Why?
13. How would you show the relationship between a graph of a function and the graph of its inverse to a student?
14. Can you illustrate what you expect the students to do? Are there other ways that are more/less appropriate? Why?
15. What is the important to teach students about the relationship between a, b and c and the graph?
16. When you receive this assignment, what was your approach to complete it?
17. Did you use any resources? If yes. Why?
18. What resources did you use for this lesson plan?
19. Why did you choose these examples?
20. Tell me about your proposed instructional sequence?
21. What do you think about your teaching of the lesson plan?
22. How did the lesson go?
23. Were you able to achieve the goals of the lesson plan?
24. How do you think the students respond to your questions?
25. How was the communication between you and the students?
APPENDIX-I
CONSENT FORM

Letter of Consent for Adults

Dear Students:

I am a graduate student under the direction of Professor Jakubowski in the program of Mathematics Education in the Department of Middle and Secondary Education at Florida State University. I am conducting a research study to understand and analyze prospective secondary teachers’ knowledge of concept of functions and how prospective secondary teachers’ knowledge interacts with their ideas about the teaching and learning mathematics and their ideas about pupils and the teaching context. In addition, this research study is designed to understand and analyze participating prospective teachers’ changes in content knowledge, pedagogical knowledge for teaching. In order to determine differences in pedagogical knowledge and enactment of this knowledge, participating teachers will be involved in different activities.

Your participation will involve interviews on concept of function questions, submitting lesson plans, keeping journals and observation and videotapes of small group teaching, observation of your MAE 4330 class.

You must be 18 years old to participate in this study. Your participation will involve ten interviews that are expected to last no more than one hour each and will include questions about concept of functions. These will be arranged at your convenience, and occur at twenty points of time during the study.

Your participation in this study is voluntary. If you choose not to participate or to withdraw from the study at any time, there will be no penalty. It will not affect your grade. Your privacy will be protected to the extent allowed by law. The results of the research study may be published, but your name will not be used to protect your privacy.

There are no foreseeable risks or discomforts if you agree to participate in this study. I request your permission to audiotape and videotape the interviews. The tape will be kept in a locked cabinet in 219 MCH for which only I or my professor has the key, and all tapes will be destroyed by March 31, 2007.

If you have any questions concerning this research study, please call me at 644 8433 or e-mail me at gh02@fsu.edu. You can also call my professor, Dr. Jakubowski at 644 8428 or e-mail her at jakubow@coe.fsu.edu If you have any questions about your rights as a subject/participant in this research, or if you feel you have been placed at risk, you can contact the chair of the Human Subjects Committee through the Vice Presidents for the Office of Research at (850) 644-8633.

Sincerely,

Signature

GuneY Haclomeroglu

I give my consent to participate in the above study.

Name:............................................................ Date:.................................

Signature:........................................................................................................
APPENDIX-J
HUMAN SUBJECT APPROVAL LETTER

Office of the Vice President For Research
Human Subjects Committee
Tallahassee, Florida 32306-2763
(850) 644-6873 · FAX (850) 644-4392

APPROVAL MEMORANDUM

Date: 9/22/2005

To: Gunes Haciomeroglu
368 Pennell Circle #2
Tallahassee, FL 32310

Dept.: MATHEMATICS

From: Thomas L. Jacobson, Chair

Re: Use of Human Subjects in Research

Examining prospective secondary teachers' subject matter knowledge and pedagogical content knowledge of concept of functions.

The forms that you submitted to this office in regard to the use of human subjects in the proposal referenced above have been reviewed by the Secretary, the Chair, and two members of the Human Subjects Committee. Your project is determined to be Expedited per 45 CFR § 46.110(b) 9 and has been approved by an accelerated review process.

The Human Subjects Committee has not evaluated your proposal for scientific merit, except to weigh the risk to the human participants and the aspects of the proposal related to potential risk and benefit. This approval does not replace any departmental or other approvals, which may be required.

If the project has not been completed by 9/20/2006 you must request renewed approval for continuation of the project.

You are advised that any change in protocol in this project must be approved by resubmission of the project to the Committee for approval. Also, the principal investigator must promptly report, in writing, any unexpected problems causing risks to research subjects or others.

By copy of this memorandum, the chairman of your department and/or your major professor is reminded that he/she is responsible for being informed concerning research projects involving human subjects in the department, and should review protocols of such investigations as often as needed to insure that the project is being conducted in compliance with our institution and with DHHS regulations.

This institution has an Assurance on file with the Office for Protection from Research Risks. The Assurance Number is IRB00000446.

Cc: Elizabeth Jankowski
HSC No. 2005.679

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REFERENCES


BIOGRAPHICAL SKETCH

Guney Haciomeroglu was born February 15, 1979 in Kahramanmaras, Turkey. She graduated as a physics teacher from Selcuk University with a Bachelor of Science degree in Physics Education Turkey in 2000. Then, in year 2001, she had come to the U.S to attend to CIES (Center for Intensive English Studies) for a year. In 2002, she had worked in Cram School in Ankara, Turkey. In summer of 2002, she was awarded with scholarship to pursue a Ph.D in the field of Mathematics Education. She started her graduate studies at Florida State University in 2002 and earned her Master of Science degree in Mathematics Education prior to entering the doctoral program. During her education at Florida State University, she has worked in the mathematics education program as a teaching assistant for online courses, research assistant for three grant projects and supervisor for teacher candidates.

Her research interests are prospective teachers’ subject matter knowledge, pedagogical content knowledge, algebraic thinking, prospective teachers’ experiences with tutoring and teachers’ metacognition. Upon finishing her doctoral degree, Guney Haciomeroglu will teach in Department of Secondary Science and Mathematics Education at Canakkale Onsekizmart University in Turkey.