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Effects of Spatial Visualization and Achievement on Students' Use of Multiple Representations

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COLLEGE OF EDUCATION

EFFECTS OF SPATIAL VISUALIZATION AND ACHIEVEMENT ON
STUDENTS' USE OF MULTIPLE REPRESENTATIONS

BY

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This thesis is dedicated to my husband
Ramazan Erbilgin
whose love and support have gotten me to this point.

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ABSTRACT

Recently, there has been a growing interest in research on students' use of multiple representations in mathematics education. This study focused on how spatial ability and achievement affect students' use of multiple representations. The methodology used was case studies. The researcher conducted 16 interviews with four 8th grade students from the same regular mathematics class: one high achieving-high spatial ability, one high achieving-low spatial ability, one low achieving-high spatial ability, and one low achieving-low spatial ability. The students were asked linear equation and function problems requiring the use of different representations. Additionally, the mathematics class was observed for 7 hours. The Wheatley Spatial Ability test was applied to the class of 8th graders to determine the spatial ability levels of the students. The students' achievement levels were determined from students' Florida Comprehensive Assessment Test scores, linear equation class exam scores, and consultation with the teacher. The findings suggest that both achievement and spatial visualization has effects on students' use of multiple representations.

CHAPTER 1

Introduction

Mathematics education has been experiencing a reform from procedural teaching and learning to conceptual and procedural teaching and learning in recent decades. There are many new curricula to implement new ideas for improving mathematics education such as *Connected Mathematics* (Lappan, Fey, Fitzgerald, Friel & Phillips, 1997), and *Mathematics in Context* (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1997/98). Some process standards such as connection, communication, reasoning and proof, problem solving, and representations employ a great importance in these new efforts, as called for in the National Council of Teachers of Mathematics [NCTM] *Principles and Standards for School Mathematics* (NCTM, 2000). This study focuses on one of these process standards: using multiple representations in mathematics education.

Rationale and Theoretical Perspective

Multiple Representations

NCTM (2000) suggests in the representation standard that students from pre-kindergarten to 12th grade should be able to do the following: "create and use representations to organize, record, and communicate mathematical ideas; select, apply, and translate among mathematical representations to solve problems; and use representations to model and interpret physical, social, and mathematical phenomena" (p.66). Another related process standard, the connection standard involves "recognizing and using connections among mathematical ideas; and understanding how mathematical ideas interconnect and build on one another to produce a coherent whole" (p.63). Therefore, if they are

interpreted together, it could be said that while teaching a concept, mathematics education should include using different representations at the same time in order to have students make translations between them and construct meaningful learning.

Traditionally, representations of a concept are taught separately, for instance, when teaching quadratic functions, symbolic analysis and graphical analysis are taught in different subunits as if they are not related. Another example might be the teaching of linear equations and drawing a line. They are taught as if they were separate concepts and skills. Moreover, in traditional classrooms, topics tend to be discussed by using a single representation. Students are not required to connect different representations of a topic.

There has been a growing research base about students' use of representations in mathematics education. Research results suggest that using multiple representations in mathematics education enhances deep understanding of concepts, improves problem solving ability, and strengthens the learning process by providing mutual sources of information (Borba & Confrey, 1993; Yerushalmy, 1997; Brenner, Mayer, Moseley, Brar, Durán, Reed, & Webb, 1997; Swafford & Langrall, 2000; Porzio, 1999; Ozgun-Koca, 1998). Moreover, Pape & Tchoshanov (2001) suggest the cognitive capacity of the human brain is aligned with multiple representational patterns. Therefore, students' thinking skills are developed if they study in a multiple representational environment. Greeno and Hall (1997) suggest that "forms of representation need not be taught as though they are ends in themselves. Instead they can be considered as useful tools for constructing understanding

and for communicating information and understanding”
(p.362).

Representation means the act of capturing a concept or relationship and the relationship itself (NCTM, 2000). This definition integrates definitions for internal and external representations. Internal representations occur in the mind of the learner and can be referred to as cognitive models, schemas, concepts, or mental objects. External representations are embodiments of mathematical ideas or concepts such as algebraic symbols, tables, graphs, verbal statements, and concrete materials (Aspinwall, 1995). Pape and Tchoshanov (2001) argue that there is an ongoing interaction between internal and external representations within the social environment.

In this study, multiple representations refer to external representations (more than one form) that provide the same information about a concept simultaneously. However, it is assumed that there is an ongoing influence between internal and external representations, where the external representation is created by the learner (not just copied from another student or teacher without making an effort to understand).

Mathematicians' views of the relationship between internal and external representation vary. Wileman (1980) accepts external representation to be the same as internal representation. According to him, what we sketch is same as what we visualize in our minds and we readily understand a good constructed visual representation. Mitchell (1994), as cited in Pape and Tchoshanov (2001), also argues that there is no difference between internal and external representation. On the other hand, Arnheim (1969) argues that external representations are different than internal

images. For example, what we draw on paper may be different than our mental images and what we see may be different than the real image. McKim (1972) suggests that using representations to develop students' thinking skills is related to students' ability to operate with mental images (for example visualizing external representations). In this study the second view is accepted. It is reasonable to argue that there is a relation between the use of representations and visualization. Thus, for this study there is a need to clarify what visualization is, in particular I am interested in spatial visualization.

Spatial Visualization

Spatial visualization has been defined in many ways. It is going to be assumed that visualization is the same as mental imagery. In this study, spatial visualization is defined as the ability for mentally rotating and comparing two dimensional objects as measured in The Wheatley Spatial Ability Test (Wheatley, 1978/1996). Wheatley (1998) describes three components of imaging: constructing an image, re-presenting an image, and transforming an image. Kosslyn (1994) defines mental imagery as an internal representation. Since one meaning of representation is externalization of internal, mental abstraction, it could be argued that there should be a connection between visualization and external representation (Pape & Tchoshanov, 2001). This argument is also supported by constructivist theory. According to constructivist researchers, mathematical knowledge can not be separated from the learner. Students' interpretations of mathematical ideas are represented in their minds as internal representations and if meaningful learning is wanted, students are going to create their own external

representations. There is going to be a bridge between internal and external representations. Moreover, students should be encouraged to use a variety of modes of representations to express their own experiences and present their own ideas. External instructional representations used by the teacher or in the textbooks are valuable when they facilitate student's construction of knowledge (Aspinwall, 1995).

Some of the profound thinkers in history stated that they think in terms of mental images (Aspinwall, 1995). Thus there are studies about the relationship between mathematics ability and spatial visualization whose results argue that spatial visual ability improves mathematical ability (Presmeg, 1986; Wheatley, 1998).

Moreno and Mayer (1999) conducted a research study to find out how achievement of low achieving, high achieving, low spatial ability (they focused on spatial visualization), and high spatial ability students changed in two different learning environments: learning by using multiple representations and learning by a single representation. They found that in a multiple representation environment, high achieving students produced a more significant pretest to posttest gain than low achieving students and similarly, high spatial ability (spatial visualization) students produced a more significant pretest to posttest gain than low spatial ability (spatial visualization) students. This study made me interested in the effects of achievement and spatial ability on students' use of representation. From my own teaching experience, I know that some students do not want to solve problems or learn a new concept with more than one representation. Is that because of their achievement level?

Or is it because of their spatial visualization level? Or are they used to learning only with one type of representation? In other words, do students with low visualization level or with low achievement level not benefit from using multiple representations? Analyzing students with different levels of ability would contribute to the instructional design of mathematical topics.

Most of the studies about using multiple representations in mathematics education are quantitative (Moreno & Mayer, 1999; Brenner, Mayer, Moseley, Brar, Durán, Reed, & Webb, 1997; Swafford & Langrall, 2000; Porzio, 1999; Ozgun-Koca, 1998). However, one of the fundamental components of a research program can be case studies to improve mathematics education (Borba & Confrey, 1993). There are benefits to studying students one-to-one. The researcher can understand a student's reasoning in depth in a clinical interview. For my research, I am going to conduct case studies to analyze students' use of multiple representations in depth. Since most of the previous studies are quantitative, I believe there is a need to develop case studies in this research area.

The mathematical concept for the study is linear functions. The reason for this choice is that linear functions offer a rich source of multiple representations. Lesh, Behr, and Post (1987) included ordered pairs, equations, graphs, and verbal descriptions of relationships in the several types of representational systems to represent functions. This study basically focused on graphs, tables, algebraic-letter representations, and verbal representations of linear functions.

Statement of the Problem

The purpose of this study was to investigate effects

of spatial visualization and achievement on students' use of multiple representations in depth. In order to achieve this purpose, this research consisted of case studies with 8th graders on linear functions. Answers for the following questions were sought:

How does spatial visualization affect students' use of multiple representations?

How does achievement affect students' use of multiple representations?

What are the effects of spatial visualization and achievement on students' use of multiple representations?

Summary

In this chapter, a rationale and theoretical perspective for the study was provided. Multiple representations have gained importance in mathematics education and the researcher believes the findings of this study will contribute to research in this area and to instructional designs in mathematics education in general.

In Chapter 2, a review of the literature is given. Chapter 3 describes the methodology used. Each student's work was analyzed and the findings from interviews and classroom observations were presented in Chapter 4. Chapter 5 discusses the effects of achievement and spatial visualization on student's use of multiple representations by comparing the cases. Lastly, Chapter 6 is about the conclusions that were drawn from this research study along with the suggestions for instruction and future studies.

CHAPTER 2

Related Literature Review

The main focus of this study is using multiple representations. Therefore, I would like to start the literature review with research and articles written about multiple representations. Since I am interested in the effects of spatial visualization on students' use of multiple representations, I will continue with the literature about spatial visualization. Lastly, I will talk about linear functions because it is the mathematical concept of this study.

What is Representation?

In mathematics, representations have been defined as internal (mental images), and external (graphs, tables, verbal, concrete, or symbolic). Palmer (1978) defines a representation as "something that stands for something else" (p.262). He discusses that a representation system involves two entities: the represented world and the representing world. Palmer (1978) further suggests that a specification of a representation should describe the following five elements: represented world, representing world, what aspects of the represented world are being represented, what aspects of the representing world are doing the representing, and correspondence between two worlds. Here, one or both of the represented world and representing world can be abstract.

Similar to this view, Vergnaud (1987) talks about three levels of entities in a representation system: the referent, the signified, and the signifier. He further contends that there are two correspondence problems between the referent and the signified, and the signified and the signifier. The referent is the real world, which is the

experience of the subject. The signified level is cognitive and it is where "invariants are recognized, inferences drawn, actions generated, and predictions made" (p.229). The signifier level consists of different symbolic systems. For instance, the syntax of algebra, graphs, tables, diagrams are different from each other. In my opinion, the signified level of Vergnaud is related to the represented world of Palmer (1978) and the signifier level is related to the representing world of Palmer.

The theoretical perspective of this study is aligned with Palmer's concept of representation system. From my perspective, Palmer's (1978) represented world is the mathematical concepts that we are trying to represent and representing world (multiple representations) is the tools with which we represent the concepts. The representing world can be internal or external. Within the representing world, I agree with the perspective of Pape and Tchoshanov (2001): there is a mutual influence between internal and external representations. They discuss a two-sided process: internalization of the external process and externalization of the internal process. They give the following example to clarify this interaction: students learn the concept of the number of objects in a set by building mental images. Then, the numeral (external representation) that represents this number of objects stimulates the mental image (internal representation).

What is the effect of using multiple representations in mathematics education?

There have been many investigations on this issue, most of which are quantitative. Porzio (1999) studied three undergraduate calculus classes: Math 151, a traditional calculus course; Math 151G, similar to Math 151 but

includes graphics calculators as an integral part of the course; Math 151C, an electronic course Calculus & Mathematica that emphasized the use of multiple representations and connections among them. The researcher made classroom observations. Students took a posttest and 12 students from each course were interviewed. The posttest involved problems requiring ability to work with graphical, analytic, and numerical representations and also could be solved by more than one representation. Math 151C students provided acceptable responses more often than the students from the other two classes did. Math 151G students provided acceptable answers slightly more often than Math 151 students did. Traditional students had difficulty in recognizing connections between graphical and symbolic representations. For me, it was interesting that the graphics calculator students did not do well on the posttest. However, data from classroom observations revealed that instruction in this class did not provide students enough time and opportunity to make connections between different representations by themselves. This finding is aligned with constructivist learning theory since it suggests that students should make internal connections themselves. Compared to the other two classes, during the interviews, Math 151C students used both symbolic and graphical representations showing a well-connected internal network.

Another study favoring the use of multiple representations in mathematics education was done by Brenner, Mayer, Moseley, Brar, Durán, Reed, & Webb (1997). They created an experimental unit on functional relationships that used graphs, tables, verbal explanations, and symbolic expressions. They studied 128

seventh and eighth grade students. Seventy-two of them attended classes that used the representation-based unit and 56 of them learned from a conventional textbook on algebra. Students took pretest and posttest. The treatment group did significantly better than the control group ($F(1,121) = 18.35, 3.74, MSE = .04, p < .05$) on problem representation and functional representation skills (no significant effect for the teacher). Even though students in the treatment group did not spend much time on solving equations, the groups did not differ on word problem solving and equation solving. With qualitative analysis of the students' responses to posttest problems, the researcher concluded that the treatment group had a better understanding of functional relationships.

A very similar study was conducted by Moseley and Brenner (1997). They studied 15 experimental and 12 control group students to examine students' abilities to work with algebraic variables and their notations as a result of the instruction they received. Students studied experimental and control curriculum respectively. The experimental curriculum emphasized problem representation skills and asked students to use multiple representations throughout the unit. Pretest and posttest interviews were done with all students. Results indicated that the control group showed an arithmetic perspective in their mathematical reasoning whereas the experimental group showed a more algebraic perspective in their mathematical reasoning.

Brenner, Herman, and Ho (1999) compared the problem representation skills of students from the United States, China, Taiwan, and Japan. The Third International Mathematics and Science Study (TIMSS) revealed the higher mathematics success of Japanese and other Asian students

(Beaton, Mullis, Martin, Gonzalez, Kelly, & Smith, 1996). The study of Brenner et al. included 895 sixth-grade students from the four nations. They developed a test measuring the ability to use multiple representations and being flexible in moving across different representations. They found that Asian students outperformed American students on most of the test items. The difference might be due to instructional materials used in each country. For instance, Japanese textbooks emphasize coordination of multiple representations; they integrate verbal, pictorial, and symbolic representations in the explanations more than American textbooks (Mayer, Sims & Tajika, 1995). Further study is required to show the effects of using multiple representations in instruction.

There are also studies focused on a single representation and benefits of that specific representation. For example, Hines, Klanderma, & Khoury, (2001) studied 8th grade students' and elementary and middle school pre-service teachers' use of tables. The eighth graders worked with a dynamic physical model whereas pre-service teachers solved real world problems to develop meaning for functions. They were guided to create tables to record the data. For both groups, tables helped them reflect upon and reorganize their thinking of functions, and grasp the relationship between variables. The investigation with middle school students also suggests that students created a link between concrete and tabular representations. This might have added more understanding about the concept of functional relationship.

Bell and Janvier (1981) studied the interpretation of graphs with two first year secondary classes composed of 12 year old students. Students working in groups conducted

four experiments (heating up water, cooling boiling water, stretching rubber bands, and filling bottles). Students in one class plotted the data whereas the students in the other class created tables from the data. The table group did better on interpolation problems while the graph group did better on fastest rate of change (slightly better), graphical awareness, and steadiness problems. The researchers suggest that the use of tables should be included in the teaching of graphs. I want to note an interesting finding in this study. They discovered that class discussions at the end of the experiments did not have all pupils reflect on their thinking and as a result, the researchers prepared synthesis exercises. This is aligned with Porzio's (1999) finding about the role of constructive instruction on deep learning. Only involving multiple representations in the class is not enough, how they are involved makes a difference.

In conclusion, the investigations in this section support using multiple representations in mathematics education. The findings show that students become better problem solvers, understand concepts deeper by forming internal connections, and improve mathematical thinking in a multiple representational environment. In fact, there have been many articles written on this issue. Meyer (2001) suggested that students can bridge the gap between concrete and increasingly abstract levels through the use of diagrams, models, tables, or symbolic expressions. Friedlander and Tabach (2001) contend that each representation has advantages and disadvantages. Using various representations can cancel out the disadvantages since students will learn to choose appropriate ones for a problem, and since they learn to represent a situation in a

variety of ways, they will become flexible in using representations. Furthermore, each individual can find out his/her own style of problem solving strategies.

I agree with the suggestions in the conclusion part of the previous section. We should give students opportunities to make transitions between different representations to enhance learning. I also would like to mention that constructivist theory should be guiding us in our teaching since students learn better when they make connections between new and previously learned concepts and between different representations of the concept.

Spatial Visualization

Haeker and Ziehen (1931) as quoted in Presmeg (1986) define three types of people with respect to mathematical abilities:

1. Subjects for whom the visual element is dominant: abstract problems are solved visually;
2. Subjects for whom the abstract element is dominant: geometric problems are solved logically, abstractly;
3. Subjects for whom the two elements are in equilibrium. (p.299)

Krutetskii wrote that "... logical reasoning and spatial notions are both necessary in mathematics" (p.117) (cited in Presmeg (1986)).

Based on previous experiences, the researcher believes that visual thinking is less encouraged than logical reasoning in most of today's mathematics classes. Moreover, the researcher believes that there is a connection between the use of multiple representations and visual ability and that internal representation means mental images. Therefore, one focus of this study is to find out how

different visual ability students use multiple representations.

I will assume that mental imagery and visualization have the same meaning. It has been difficult to define mental imagery since it is inherently internal (Aspinwall, 1995).

Presmeg (1986) defined a mental image as "a mental scheme depicting visual or spatial information" (p.297). Sweller (1994) defined that "a scheme is a cognitive construct that organizes the elements of information according to the manner with which they will be dealt" (p.296). He also claimed that effective schemes are going to increase the amount of information to be held in the working memory.

Wheatley (1998) discusses imaging as a mental activity consisting of three components: constructing an image, re-presenting the image, and transforming the image. During the daily activities such as solving mathematical problems, people construct images. For example, if students are shown a geometric figure briefly and asked to draw it, they will draw it from the constructed image. However, what a person constructs depends on his image schemes. In other words, we are affected by our experiences.

Wheatley's (1998) second component, re-presenting the image, is about re-calling an image after it was constructed. A re-presented image may not be the same as the original constructed image. It is influenced by the goals of the re-presenter at the time of re-presenting.

The third component, transforming the image, is a dynamic process. It is about changing the position of the image or the re-presented image such as rotating, or folding. Wheatley (1998) gives mathematical examples for

transforming an image, when constructing $y=3\sin(x)$, we transform $y=\sin(x)$. In adding $7+5$, we may transform 5 as $3+2$ and combine 7 and 3.

Wheatley (1998) concluded that mental imagery has an essential role in all kinds of mathematical activity as a result of many clinical interviews. He presented a set of chips to primary level students and asked them to count the chips. Then he placed some of the chips in his hand, closed it, and asked students how many chips there were in his closed hand while showing the remaining chips. Afterwards, students were asked to draw circles (as the number of total chips) on their paper and the task was repeated. Some number of chips was replaced on the circles and the number of the remaining chips was asked. With this presentation of the task students mostly answered correctly. Moreover, when the paper was no longer shown, students did better than previously. Some were drawing circles in the air. This activity suggests visualization helps students perform number operations. Presmeg (1986) suggested that imagery can be useful for students to make generalizations in mathematics in two ways; concretizing the referent, and using pattern imagery.

Paivio (1971), known as the founder of the dual-coding theory, contended that one stimulus composes both a linguistic system and an imagery system providing two memory traces. The two systems are interconnected even though they are independent. A person can translate non-verbal information into verbal and verbal into non-verbal information. Paivio (1971) further discussed that we connect verbal and visual representations during the meaning making process. This is the rationale for the use of multiple representations. If students learn a concept

with more than one representation, they can make connections between them and improve mental schemes, which results in better learning.

Now, I will turn back to Moreno and Mayer's (1999) study that piqued my interest in the effects of achievement and spatial ability on students' use of representation. They conducted a research study to find out how achievement of low achieving, high achieving, low spatial ability (they focused on spatial visualization), and high spatial ability students changed in two different learning environments: learning by using multiple representations and learning by a single representation. The study was composed of two experiments: one with 60 and the other with 26 sixth grade students. Students studied number lines with computer-based material. They found that in a multiple representation environment, high achieving students produced more significant pretest to posttest gains than low achieving students and high spatial ability (spatial visualization) students produced more significant pretest to posttest gains than low spatial ability (spatial visualization) students. In their study, they concluded that high achieving and high spatial ability students benefit from multiple representation environments most and it does not make a significant difference in the learning of low achieving and low ability students. This study was quantitative. However, I want to observe different ability level students' use of multiple representations in depth by studying four 8th grade students. In addition, the topic will be different. The mathematical topic of this investigation is going to be linear equations. I believe the findings of case study research will give deeper insights about this issue and will contribute to

mathematics education by suggesting instructional designs.

Linear Functions

The rationale for linear functions as the mathematical concept of this study is that linear functions can be displayed by different representational systems. The teaching of linear functions usually involves use of multiple representations such as tables, graphs, and equations. The researcher sought to ask questions and pose problems involving the use of multiple representations in this area and to observe how different spatial ability and achievement level students used multiple representations through the problem solving process. Another reason for the choice of linear functions is that it is a central topic in the algebra curriculum (O'Callaghan, 1998). There has been a growing research literature around this topic (Lesh, Behr, & Post, 1987).

Representations can take different forms from verbal to symbolic (Swafford, & Langrall, 2000). Narrative descriptions of a relationship are the verbal representation of the relationship. Verbal representations are usually used in the problem posing or in the interpretation of the results. Numerical representations are the systematic organization of specific cases. They can be effective bridges from arithmetic to algebra (Friedlander, & Tabach, 2001). Graphical representations are effective in providing a clear picture of the relationship. Symbolic representations use variables to present a general and concise pattern.

Verstappen (1982) as cited in Kieran (1992), states three ways to define a linear function in mathematical language: 1) geometric-schemes, diagrams, histograms, graphs, drawings; 2) arithmetical-numbers, tables, ordered

pairs; and 3) algebraic-letter, symbols, formulas, mappings. Lesh, Behr, and Post, (1987) included ordered pairs, equations, graphs, and verbal descriptions of relationships in the several types of representational systems to represent functions.

The present study will focus on verbal, graphical, numerical, and symbolic representations of linear functions. Verbal representation will refer to any narrative descriptions that students use for interpreting the linear functions. Numerical representations will be referred to primarily as tables. Graphical representations will refer to lines or points graphed on the coordinate axes. Symbolic representations will be referred to primarily as linear equations.

O'Callaghan (1998) defined four components in a function model: modeling, interpreting, translating, and reifying. He described modeling as representing a problem situation by any mathematical representational system. The reverse procedure of modeling is interpretation. Translation refers to moving from one representational system to another representational system. The fourth component of his function model is reification, which means the creation of a mental object from formerly learned processes or procedures.

Leinhardt, Zaslavsky, and Stein, (1990) analyzed the concept of functions, graphs, and graphing in four components: the action of the learner, the situation, the variables and their nature, and the focus. I will only describe the action of the learner since I am interested in students' work. Leinhardt et al. described four typical tasks that make up the action component: prediction tasks, classification tasks, translation tasks, and scaling tasks.

By prediction they referred to conjecturing other parts of a graph where some parts of it are given or a rule where some number pairs are given. Prediction tasks may require estimation, measurement, or pattern detection skills. In their study, classification referred to deciding if a given relationship is a function or not, recognizing a functional relationship among other relations, and recognizing special kinds of functions among other functions. By translation, they referred to identifying the same function when represented by different representations, identifying the corresponding transformation of a function in another representational system when its transformation is given in one system, and constructing a function in another representational system when it is given in one system. Lastly, scaling is about the scale of the axes and the units that are used in the question.

My theoretical framework for linear functions is based on the studies of Leinhardt, Zaslavsky, and Stein, (1990), O'Callaghan (1998), and my own purposes for this study. In this study, the work of students about linear functions will be analyzed in four areas: translation, classification, interpretation, and preference and tendency. Different from the other two studies, my study includes preference and tendency because I am interested in how different achievement and spatial ability level students use multiple representations, including what representation each student prefers and what representation each student uses more frequently. Translation will refer to constructing a representation of a linear function when it is given in another representation. Here different representation systems will include verbal, symbolic (equations), numerical (tables), and graphical

representations. Classification will refer to identifying specific types of linear functions such as increasing, decreasing, or constant. Interpretation will refer to the actions by which a student makes sense of the linear relationships (such as how they explain what $y=5$ means, or how they compare two or more linear relationships). Preference will refer to the choice of representation of a student when he/she is given more than one representation. Tendency will refer to what representation a student uses more often in answering questions and solving problems dealing with linear functions.

This chapter included the literature review related to my study: literature about multiple representations, literature about spatial visualization, and literature about linear functions. This review provides a strong base for the design of my study, the data collection, and the analysis of the collected data.

CHAPTER 3

Methodology

As stated in Chapter 1, to understand students' use of multiple representations in depth, the case study methodology was selected for this investigation. Clinical interviews were done with four 8th grade students.

Participants

Four students were chosen by using purposeful sampling. I formed four categories as the basis for my selection: high achievement-high spatial ability, high achievement-low spatial ability, low achievement-high spatial ability, and low achievement-low spatial ability. For each category, I selected one student. Moreover, I tried to select talkative students because I was interested in their thinking processes and I wanted students to express what they thought. A quiet student would provide limited information about his/her thinking process.

There were 30 students in the 8-E class in The Florida State University School. The class was a regular 8th grade mathematics class. The school also had an 8th grade algebra class. The students in the 8-E class used *Middle School Math Course 3* (Foresman & Wesley, 2002) as their mathematics textbook.

The procedure for student selection was as follows. Twenty-nine of the students in this class took the Wheatley Spatial Ability Test [WSAT] (Wheatley, 1978, 1996). The spatial ability levels of students were determined according to the test results. The details of scoring will be explained in the instruments section. To determine the achievement level of students, I analyzed the students' linear equation exam scores from their mathematics class, and Florida Comprehensive Assessment Test [FCAT] scores

from the 7th grade. The students had the same linear equations exam twice. I used both scores. FCAT has two sections of math and reading tests for all students in grades 3-10: Criterion-referenced Test (assesses students' achievement as represented in the Sunshine State Standards) and Norm-referenced Test (compares the performance of Florida students to the Math and Reading performance of students across the nation) (Florida Department of Education, 2003). Therefore, students had two kinds of FCAT scores for mathematics. In addition to these scores, I consulted with the classroom teacher about who was high achieving and who was low achieving as well as who was talkative and who was quiet. The researcher and her academic advisor, Dr. Maria L. Fernández, analyzed all the data about achievement and spatial ability level of the students and selected students as follows: one high achieving-high spatial ability student, one high achieving-low spatial ability student, one low achieving-high spatial ability student, and one low achieving-low spatial ability student. All four students were about age 13 or 14. None of them was repeating the 8th grade.

Instruments

The Wheatley Spatial Ability Test [WSAT] (Wheatley, 1978, 1996) was used to determine students' spatial visualization level. It was developed in 1978 and widely used as a predictor of students' mathematical potential. It is valid and reliable with a high internal consistency (K-R=.92). The WSAT is a 100-item pencil and paper test designed to measure the ability to mentally rotate and compare figures. Students are given a figure at the left on the page. For each item, they are required to decide if this figure will match each of the five figures to the

right by turning the original figure in the plane of the page or flipping to match. Answers are recorded directly on the test papers to eliminate any other factor influencing results. There have been students who scored highly on the WSAT but have not done well in school mathematics (Wheatley, 1978, 1996). This suggested a rationale to find four students with different compositions as explained in the participant section above.

The test is timed: 7 minutes for 7th graders. Wheatley (personal communication on email) indicated that 7 minutes was also appropriate for 8th graders. The reason for timing is to discourage analytic strategies in responding to items. Students are not expected to complete the test. However, there have been studies where women responded slower than men on spatial ability tests (Birenbaum, Kelly, & Levi-Keren, 1994). Goldstein, Haldane, and Mitchell (1990) showed that men did not score higher than women in an untimed spatial ability test (Vandenberg Test) or when assessment was done in terms of ratio-scores (number correct/number attempted). Considering these findings, I told students to put a sign on the question they were doing, 7 minutes after the test had begun and then I let them finish the test. I assessed the test in two ways: as described in the WSAT and as ratio-scores. For the first method that is the procedure described in the WSAT, I only counted the questions answered in the first 7 minutes. The scoring for this is $\text{Number of correct answers} - (1/2) * \text{Number of incorrect answers}$. To find the ratio-scores, I took into account all the questions answered. I divided the number of correct answers by the number of questions attempted. I determined the spatial ability levels of students by considering both scores. For example, to define a student

as a high spatial ability student, I made sure that his/her two kinds of scores were high. Therefore, the two high spatial ability students who were interviewed had high scores in the two kinds of scoring and the two low spatial ability students who were interviewed had low scores in the two kinds of scoring.

Another instrument for this study was the interview questions. I conducted four interviews with each student (making 16 interviews in total). The questions in interview 1 are given in Appendix A, interview 2 is given in Appendix B, interview 3 is given in Appendix C, and interview 4 is given in Appendix D. Interview 1 includes questions to observe how students translate linear functions from one representation to another. Interview 2 includes translation and interpretation questions. Part-c of the second question in the second interview was asked to learn about students' preferences and tendency for representations. The questions in interview 3 were asked to observe how students classify linear functions and which representations they prefer and tend to use in the problem solving process. Interview 4 includes interpretation and classification problems. However, I would like to mention that although each interview was designed to observe certain abilities of students as described above, I was able to gathered data on students' understanding of all four categories of linear functions from each interview.

Besides interviews, I conducted classroom observations. Notes from the classroom observations, worksheets that students studied in the class, and the textbook that the class was using were used in the preparation of questions for the interviews and in the data analysis process. The worksheets are given in Appendix E.

Procedure

Data Collection and Analysis

Four students were chosen according to the four compositions defined in the participation section. The researcher conducted 4 interviews with each of them separately (making 16 interviews in total). The researcher frequently reminded students to think aloud. Each interview took about 45 minutes and included about 3 or 4 problems. Interviews were videotaped and transcribed. Video-taping allowed for better recall of representations used by the students. Students' written work from the interviews were collected and used in the data analysis process.

In addition to clinical interviews, I observed the class to become familiar with the students and their learning environment. I observed the class for 5 class periods before the interviews and 2 class periods during the interviews. Out of these 7 periods, students studied linear equations for 2 class periods (before the interviews), and linear functions for 2 class periods (during the interviews). The timeline for the interviews and class observations was as follows. The students learned about linear equations in November 2002 (fall semester): during this time, five classroom observations took place. The students learned about linear functions in April 2003 (spring semester): during this time, two classroom observations were completed. Three of the interviews were conducted in March 2003. The fourth interview was conducted in the second week of April 2003 after the students had learned about linear functions in the class.

I analyzed the linear equations chapter of the textbook that the students were using and the exercise sheets that students completed in the class to get familiar

with the kinds of questions they were working on in the class. This helped me to prepare or select appropriate questions for the students' grade level.

During the interviews, students were required to solve problems involving the use of multiple representations. I tried to understand which students were better at what kinds of representation, if achievement and spatial visualization had an affect on students' preferences of representations when the problem was first asked and during the problem solving process, which students could make connections between representations, and how deep each student's understanding was with respect to multiple representations of linear equations.

For instance, one problem involved four children's savings throughout the year. Each child's savings is given with a different representation: table, graph, equation, and verbal. Students were asked to find the savings of each child at the end of the 8th and 40th weeks. Then, they were asked to choose two children and compare their savings using tables, equations, or words like "the savings increases (or decreases)", "the savings increases (or decreases) at a rate of ...", and "who has a larger (or smaller) amount at the beginning (or end) of the year. Another question in this problem required describing each child's savings. This problem provided insight about how students operated with representations, how they made connections within representations, and what their preferences for representations were.

For the first three interviews, I prepared the problems before I started interviewing because I wanted to pose certain problems in order to gather specific information. After the third interview, I transcribed each

interview and summarized my findings about each student. I tried to find out whether I needed more information for each student or not. I made a list that showed what information I was missing for each student. Then, I prepared a list of questions for each student and conducted the fourth interviews. Therefore, the first three interviews included the same questions for each student but the fourth interview consisted of common as well as some different questions depending on the student. After the fourth interviews, I transcribed them and summarized the findings for each student. Then, I felt that I had enough information for each student. Therefore, I stopped interviewing. I organized the findings as follows. I made a table for each category that I created to analyze linear functions: translation, classification, interpretation, and preference and tendency (so I had four tables in total). Each table was designed to have one column for each student (making four columns). These tables facilitated the comparison of students with each other.

Lastly, I used the four tables, the summaries for each student, the transcriptions of the interviews, students' written work from the interviews, and the notes from the classroom observations to write the findings and the conclusion chapters.

Limitations of the Study

For this study, the researcher conducted 16 clinical interviews with 8th grade students (4 interviews with each student). In addition to this, the classroom was observed for 7 class periods. I believe that one of the limitations of the study was that the researcher was inexperienced in making clinical interviews and analyzing what is going on in students' minds. To minimize this factor, before the

first interview, I interviewed a middle achieving 8th grade student and modified some of the problems. Reading research articles and books about case study research also helped me to become more confident in completing a case study investigation. However, I learned a lot about natural inquiry during the interview process. I recognized that practice is one of the necessary components to become an expert in conducting case study investigations. Another limitation might be students' interest in the study. Sometimes students got bored. To eliminate this possibility, I observed the classroom and helped students in doing their exercises in the classroom. I talked to them about their family life to make our relationship friendlier. They became familiar with me, and fortunately no one gave up interviewing.

CHAPTER 4

Findings

In this chapter, I present the analysis of the clinical interviews that were conducted with four 8th grade students, and the classroom observations. First, I will briefly describe each student's achievement and spatial visualization scores along with class averages and ranges (see Table 1 for a summary of the achievement and spatial visualization scores for the four students in this study and their class averages and ranges). Then, I will present the findings of each student separately.

Table 1: Achievement and spatial visualization scores

	WSAT SCORE	RATIO SCORE	FCAT LEVEL	FCAT NPR	CLASS EXAM I	CLASS EXAM II
David (LA-LV)	54.4	0.89	2	52	12	22
John (LA-HV)	95.5	0.97	2	62	18	11
Lara (HA-LV)	26	0.82	3	81	9	24
Rose (HA-HV)	97.5	0.989	4	97	19	28
Class Average	69.8	0.94	2.8	78.7	15	20
Class Range	25-97.5	0.82-1	1-4	14-97	9-21	8-29

*For an explanation of the abbreviations read the text on the following page.

To present a clearer picture to the reader about the scores of the four students, I will first give the class average and range for each kind of score. There were two kinds of spatial visualization scores for each student: one was calculated as described in the WSAT and the other was ratio-scores. The class average in WSAT was 69.8. The scores ranged from 25 to 97.5. The class average of ratio

scores was 0.94 and the scores ranged from 0.82 to 1. For the criterion-referenced mathematics test section of the FCAT, I got the achievement levels (1-low to 5-high) of the students. The class average was 2.8 (the state average was 2.4) and the students had levels from 1 to 4: 11% of the students were in level 1, 14% of the students were in level 2, 57% of the students were in level 3, 19% of the students were level 4. For the norm-referenced mathematics test section of the FCAT, I got the National Percentile Rank [NPR] scores for all students. The class average was 78.7, and the median score was 81. The median score for the NPR was 74 in the state (Florida Department of Education, 2003). The students had percentile scores from 14 to 97. The class averages for the two linear equation class exams (they included the same questions) were 15 and 20 out of 31 possible points.

David was chosen as the low achieving-low spatial ability (LA-LV) student. His WSAT score was 54.5 and his ratio score was 0.89. His mathematics score for FCAT was at level-2 and his NPR for mathematics on FCAT was 52. He got 12 and 22 out of 31 possible points on the two class exams about the linear equations chapter. John was selected as the low achieving-high spatial ability (LA-HV) student. His WSAT score was 95.5 and his ratio score was 0.97. His mathematics score for FCAT was at level-2 and his NPR for mathematics on the FCAT was 62. He got 18 and 11 out of 31 on the two class exams about the linear equations chapter. The mathematics teacher defined both David and John as low achieving and talkative students.

Lara was chosen as the high achieving-low spatial ability (HA-LV) student. Her WSAT score was 26 and her ratio score was 0.82. Her mathematics score for FCAT was at

level-3 and her NPR for mathematics on the FCAT was 81. She got 9 and 24 out of 31 from the two exams (the exams were exactly the same) about the linear equations chapter. Rose was chosen as the high achieving-high spatial ability (HA-HV) student. Her WSAT score was 97.5 and her ratio score was 0.989. Her mathematics score for FCAT was at level-4 and her NPR for mathematics on the FCAT was 97. She got 19 and 28 out of 31 on the two exams about the linear equations chapter. The teacher defined both girls as high achieving and talkative.

The Case of David (Low achieving-low spatial ability)
Translations

I will describe the findings on translations in four sections: translations to graphs, translations to tables, translations to verbal representations, and translations to equations.

Translations to graphs. David could translate tables to graphs easily. He could plot the pairs in a table (as points). He was able to plot the table in problem-1 (Appendix A) on a coordinate grid. He could also plot the line for the linear equation $y=-4x+1$. He used a table to go from the equation to the graph. He only used positive numbers. The 4th question in the first interview gives the slope of a linear equation and asks to sketch the graph. David sketched the graph by thinking run and rise, however, he only made points. This reflects his low achievement since he did not see the continuity of the equation in this situation. He again used only positive numbers. His tendency to use positive numbers is related to both achievement and visualization since operations with positive numbers are easier and making the line longer to the negative side requires visualization.

Translations to tables. David could generate a table for the graph he plotted for the 4th question in the 1st interview. He could also create tables for most of the equations. (To plot $y=-4x+1$, he created a table; to explain $y=5$, he created a table.) For the baby girl problem in the first interview (Appendix A), upon my suggestion he created a table from the verbal explanations.

Translations to verbal statements. David could interpret tables in terms of words quite well. For example for the 2nd question in the 2nd interview for Dina's savings he said: "She adds 7 dollars to her savings every week. On the 8th week she has 56 dollars and also for the 40th week she has 280 dollars." For problem-3 in the 4th interview, he interpreted Adam's savings, which was represented with a table, as decreasing by 5.

Actually, David could explain any representation in words fairly well when it was related to a context. For example, for translation from a graph to verbal statements in the 4th interview (see question-1 in Appendix D), we had the following conversation:

I: Can you compare the temperature in two cities?

D: Tallahassee increases by 1 in each month and city-J also increases temperature by 1 in each month.

I: Why did you say 1?

D: Because when it reaches the 1st month it is negative 1 and when it reaches 2nd month it gets to 0 so that is adding 1. Then when it gets to 3rd month it gets 1 degree and for the 4th month it is 2.

I: So are they increasing at the same speed or one is increasing faster?

D: At the same speed.

I: Why?

D: Because when city-T increases by 1 temperature in the first month, city-J also increases by 1.

I: Very good. They are increasing at the same speed. But the temperature in two cities is not same. What is the difference?

D: The difference is 4 degrees.

I: Why?

D: It is 3 degrees difference because I added positive 3 to negative 2 and came up with 1.

(Interview 4)

This interpretation was amazing for me because until that question he did not show a comprehensive understanding of the graphs as will be exemplified in the remaining analysis. His initial response seemed to be a computational error. Based on this observation, I began to look across David and all the other students' interviews for ways in which context affected students' understanding of the mathematical concept and representations. Analysis of David's and other students' responses helped me to see how context affected students' understanding and representations. (Examples for other students will be given in their findings sections.) In the 3rd interview I asked David if the equation $y=-5x$ (card 11, which included no context) was decreasing or increasing. He replied as relating it to a savings: "3 times negative 5 will be negative 15 because she takes 5 out each day. She takes 5 out one day, two days 10, and 3 days 15. So she will still be decreasing" (Interview 3). He was trying to make sense of the equation with a context. However, since his achievement level was low, he usually did not give detailed

information or explanation for an equation. For Danny's savings, an equation (see the second question in Appendix B), David just said it was decreasing yet for other representations he mentioned the amount of increase or decrease. To relate a pair of values from a table to the equation (part f of the 1st question in Appendix A) he said: "By multiplying by 14. So I will write 3 and 42 are related to the rule by multiplying by 14" (Interview 1). Other students' responses for the same question involved explanations of which number is x and which number is y, and plugging numbers in x and y such as $3 \cdot 15 = 45$. David's understanding of equations was not strong. This is also true for a graph given without any context. For example, for card-5 (see Appendix C), David said it is decreasing whereas for a table he prefers to mention the amount of increase or decrease.

Translations to equations. All four students had the most difficulty in this part. Actually this was found in other studies as well (Kieran, 1992). I observed two main factors in writing an equation from other types of representations: relating the representation to a context, and ability of pattern finding.

In the clinical interviews, the only question that required translation from verbal representation to any other kind of representation was question 2 in the first interview (see Appendix A). Although David understood the context of the problem, he suggested the rule was y equals 6 times x. The following conversation happened:

I: So what is happening to the baby girls after they are born?

D: They are gaining 1 pound per month.

I: And this baby girl was 6 pounds when she was born.

D: The rule is y equals 6 times x .

I: How did you come up with this rule?

D: 6 pounds when she was born would be multiplied by the months. So y would be the weight and x would be the months after she was born.

(Interview 1)

This response was probably due to the rule discussed in the previous question ($y=15x$) because in general I observed that David automatically connects a problem to another problem even though they are totally different, which reflects his low achievement. Another reason for not getting the correct answer for this problem is that David hardly ever checks his responses. Moreover, he usually does not tend to use another type of representation to help him reach the solution. For the baby girl problem, I suggested he make a table. Then, he was able to find the rule based on the table.

In the third interview, I asked him to write an equation for card 8. He wrote $15 \cdot h = 90$. Then I asked him to write a rule that he could use for any number. He wrote $h=6$. For the first question in the first interview, he could not write the equation properly from the table. Even though he recognized the pattern, he had difficulty expressing it with x and y . From the same interview, for the 3rd question he was supposed to translate the graph to an equation form:

D: Would it be subtracting by 2? Going by twos?

I: Let's write it down.

D: 1 subtract 2 would be negative 1.

I: What about 2 subtracted 2?

D: Now we would subtract 2 by 4. I mean 4 from 2; negative 2.

I: Yes, 2 subtract 4 is negative 2 but how are we going to write it as an equation?

D: I do not know this part. (Interview 1)

The examples above suggest his lack of understanding of the concept of equations. He thought of a different rule for each pair of values in the last example. He also had difficulty in using variables appropriately. He gave up easier than other students. For example, for the above problem (3rd problem in the 1st interview) all others made a table. However, David stopped trying after he said "I do not know this part." I suggested to David that he make a table. After he made the table, he could sort out the rule with my guiding questions.

If the pattern was easy, David could see it from the table such as the baby girl problem in the 1st interview. But if it was hard, he hardly ever found the rule unless it was related to a context. Two class periods that I observed covered pattern finding. In the first class they played a game called "guess my rule." As the name suggests, the class was supposed to find the rule in the teacher's mind by giving her input numbers and getting output numbers from her. David never attempted to find a rule; he was very silent that day. In the second class period, students did a worksheet about linear functions (Appendix E1). They started with rule finding from tables. By the time the other three students in this study passed to the other section of the worksheet, David could only figure out the rules $y=x+6$, $y=-2x$, and $y=9x$, which were relatively easier than the other rules $y=2x+1$, $y=3x-1$, and $y=-5x+1$.

However, he could write $y=3$ for Lisa's jogging pattern (Card-12, see Appendix C). He was also the student who found $y=5x+10$ easiest among all four students because he was reasoning about the context (Moshon's savings, see Appendix B). These examples suggest that understanding a related context helps him to find out the rule. I would like to mention that he found the rule after making a table, which I suggested for Moshon's savings.

Classification

David could classify tables according to being increasing, decreasing, or constant (Card-1, card-9, and card-10 in Appendix B). Moreover, he could identify the amount of increase or decrease such as it increases by 2, or it decreases by 3.

David's classification of graphs was very poor due to his low achievement. For card-4 in the 3rd interview, we had the following conversation:

I: Is this graph increasing or decreasing?

D: Half increasing, half decreasing. This way it is increasing because the arrow is going up, and this one is decreasing because it is going down.

(Interview 3)

For card-5 he said: "it is decreasing. It stays in negative position." He was not analyzing the graph, and he was not thinking of the pairs on the line and was not comparing the y values. However, he perfectly interpreted problem-1 and problem-2 in the 4th interview in terms of being increasing and decreasing (see Appendix D). I will discuss his interpretations in the interpretation section. I believe this difference is due to the fact that problem-1 and problem-2 in the 4th interview are related to a real

world context whereas card-4 and card-5 in the 3rd interview were presented without any context.

David's classification of algebraic equations was based on positive and negative numbers; if all the numbers are positive it is increasing, if all the numbers are negative it is decreasing, if there are both negative and positive numbers then the equation increases if the positive number is higher than the negative number, and vice versa. For $y=5$, he said: "If you multiply by 0, it will decrease; by 1 it will be constant. I think it will be increasing" (Interview 3). Then, I explained the constant equation to him but at the end of the interview, for the last question he sorted $y=5$ with increasing relationships. However, in the 4th interview, which took place after students learned about linear functions, I asked him if $y=5$ is increasing or decreasing or staying the same. He said it stays same.

When an equation was represented with a context, David was better at classification. Even though, 200 is greater than 5, he said $200-5x$ is a decreasing relationship for Danny's savings (Appendix B). Still, his explanations were not detailed. For example, for Danny's savings ($y=200-5x$) he explained that his money decreases weekly by 5 as follows: "Because he multiplies it by the number of weeks he works and subtracted by 200 and that is how he gets his money" (Interview 2). He could have given examples by substituting week numbers and comparing the results.

For verbally represented linear relationships, he did not have difficulty in understanding or classifying them. For Yonni's savings (2nd interview) he said it stays the same, for Phil (3rd interview) he said his savings

increases, and for Deanne (3rd interview), he said her savings decreases.

In the 4th interview I asked David what $y=5$ means. He said when you make a table all values are 5, and he made such a table. I asked whether they are increasing or decreasing. He replied that the equation stays the same and the table increases. He was looking at the x column of the table. Then I asked him to imagine the graph. He said it is going to be an increasing graph. When he actually drew it, he recognized that the graph stays the "same" (Interview 4). Then I asked about the table again and he said x is increasing and y is decreasing. This example shows that he does not make the conceptual connection between the different representations. He thought the equation was constant, and the table of that equation was increasing. Another observation that I drew from this example and previously discussed items (such as when he made a table to graph $y=-4x+1$, he could not graph it directly from the equation) is that his low spatial ability affected his imagination of the equation and the table as a graph. It suggested that he could not visualize the graph from the equation and the table. The last thing I would like to mention about this example is that his last response about the table that x is increasing and y is decreasing might not have happened if the question involved a context. I would like to remind the reader of his interpretation for $y=-5x$. He interpreted it correctly by relating the equation to a context.

Many times, I asked David to find the slope from different representations. First of all he never related the slope to the relationships being increasing, decreasing, or constant. Secondly, although he said the

slope is rise over run, he divided y by x to find it from a table and found different slopes for the same table and it seemed okay for him. This shows his lack of understanding of slope. From the graph sometimes he could figure the slope out by dividing rise over run but sometimes he picked any point on the graph and found the slope by dividing y by x .

When I asked David to find the slope from an equation he said he did not remember that. Whereas the other students tried to make a table or a graph from the equation to find the slope, David just said he did not remember. This is a very common behavior I recognized during the interviews. For the majority of the time, he did not tend to use another representation or another strategy to solve the problems. This was both as a result of his low achievement and low spatial ability. Achievement affected his confidence in using other representations since he did not have a deep understanding of each representation except perhaps tables, which is the only representation he rarely used as a tool during the problem solving process. Spatial visualization affected his flexibility to go from one representation to another.

Interpretation

This section is very connected to the other sections so I already wrote about some of his interpretations with respect to his translations to verbal representations.

The 4th interview included some comparison problems (see Appendix D). I showed David two tables and asked him to find which one was increasing faster. He selected the correct one and explained as follows: "This one is adding by 3, and this one is adding by 2" (Interview 4).

When comparing the two lines representing the temperature in two cities (problem-1 in Appendix D), he was the only student who correctly stated the temperature difference (3 degrees) between the two cities. Although he said the temperature increases by 1 in city-T, he found the slope as $3/2$, which shows that he was not making any connection between the rate of change and the slope. To compare the growth of the two babies (see problem-2 in Appendix D) he said:

D: Baby-A is increasing faster than baby-B.

I: Why?

D: Because baby-A increases by 1 pound and baby-B increases a half pound in each month.

I: What about at the beginning?

D: Baby-B was weighing more than baby-A, but at the end baby-A weighs more than baby-B.

(Interview 4)

He made further interpretations about this question stating when the two babies weigh the same, and how much they were weighing when they were born. Although he is a low achieving student and until now he did not show a comprehensive understanding of graphs, his comparisons of the two cities and two babies were no different than the other students. I interpret this change in his interpretations to the fact that these questions were presented in a real life context.

A last observation I want to mention about David is that he was very slow in his responses.

Preference and Tendency

To understand students' preferences I asked two questions: part-c of the 2nd problem in the 2nd interview (Appendix B) and question 7 in the 3rd interview (Appendix

C). For the first question, he chose Dina (table) and Yonni (verbal). For the second question, he picked card-6 (verbal). I interpret this in two ways: he goes by the easiest (he mentioned that he picked Dina because it is easier to figure out her savings) and his preference is verbal statements since they are related to a context and context helps him make sense of representations.

As mentioned earlier, the majority of the time during the problem solving process, he did not tend to use any other representation to reach the solution. He rarely used tables to make translations from graphs and verbal statements to equations or to explain an equation. Also, he used verbal representations to explain a relationship or other representations. For example, when he was asked to write a rule first he was stating the rule in words. He even created a context for a non-contextual question. In conclusion, I may say that he had a tendency to use tables and verbal representations.

The Case of John (Low achieving-high spatial ability)

Translations

Translations to graphs. I would like to start this section by letting the reader know that John used tables for almost any kind of translation.

He could plot the points from a table to a coordinate axes as in the 1st problem in the 1st interview. Similar to David, he plotted only points. To graph a line whose slope was given, even though I asked him to graph the line before making a table, he first made a table and then plotted the points, not the line. Even for the equation $y=-4x+1$, John plotted only points; he did not connect them to make a line (David connected them). There may be two reasons for this. The first one is his low achievement level, which affected

his understanding of an equation and graph and the connection between them. He did not consider the graph of an equation would be continuous. The second reason may be the fact that he was concentrating on the table he made for graphing the equation. By doing so, he was only considering the pairs on the table and plotting them. Janvier (1978, 1981a, 1981b) suggested that students have a point-wise focus on graphs because they make tables to plot the graphs (cited in Leinhardt, Zaslavsky, and Stein, 1990). Students typically use graphs in the same way as they use tables. John's extensive use of tables may cause his point-wise focus on graphs. (Other examples about his point-wise focus on graphs are given in the interpretation section.)

Similar to David, John also used positive numbers and plotted the points or lines to the positive side of the x axis only.

Translations to tables. This type of translation was the easiest for John. He could make a table from any kind of representation. Moreover, to go from graph to equation, verbal to equation and equation to graph he made tables (2nd and 3rd questions in the 1st interview, and 1st question in the 2nd interview). When I asked him to explain what $y=5$ meant he made a table. Tables were helping him understand any kind of representation. For example, I had a hard time explaining to him about Yonni's savings, a constant relationship represented in words. Only after I asked him to make a table was he able to understand it.

Translations to verbal statements. John could express tables in terms of words such as "Dina makes seven dollars in a week; I am guessing it increases" (2nd interview, 2nd question). When given a table, he could say "it is increasing by 2" (Interview 4).

Similar to David, John was also successful at translating graphs to words especially when the problem was contextual. In the 2nd interview for the 2nd question for Moshon's savings (graph) he said: "He is going to increase. As you look at the graph, you can see the line going up. It increases. At the beginning, he had only 5 dollars, no at zero he had 10, and at 1 he had 15 so he is increasing by 5 dollars each week" (Interview 2). Thus even to explain a graph he was constructing a table in his mind. He made a table in his mind to explain equations, too. For example, to explain that $y=-5x$ is decreasing he said: "I was just doing like negative 5 times 2 is negative 10, negative 5 times 3 is negative 15 and it just keeps decreasing" (Interview 3). He regularly used tables to make sense of other representations.

An example of his translation from equation to verbal representation is the 1st question in the 1st interview. He explained the relationship between the pair (3, 45) and the equation $y=15x$ as follows: "Once again, 45 miles is y (he writes 45 over y) and 15 miles per hour is the motorbike traveling at, and then how many hours is it traveling, 3 times 15, it was 45 miles. It was traveling 3 hours and it was going at 15 miles per hour and it was covering 45 miles in three hours" (1st interview). However, when the equation was more complex such as Linda's savings ($y=300-10x$) he preferred to say: "it decreases" without mentioning the details.

John was not very good at understanding verbal representations. In the 1st interview, he misunderstood the baby girl problem (2nd problem in Appendix A) as if they were gaining 1 pound in 6 months. Furthermore, he thought the question was asking for the baby's weight at the end of

the 6th month. (It was asking for the rule.) He was the only student who had difficulty in understanding Yonni's savings (verbal representation of a constant relationship). There may be two reasons for this. Firstly, his reading achievement in FCAT was not good (level-2). Thus he may be low achieving in reading. Secondly, I recognized that he needed to read or see something in order to understand it easily. Sometimes I wrote down my question to help him understand it even for a question such as "Can you find the y value when x is 0.5?" This seemed to help him. I was trying to explain Yonni's savings by talking only but he might have needed to read what I was saying.

Translations to equations. Among all four students John was the best at pattern finding. This might be one of the reasons why he was making tables to make sense of other representations. The study of Hines, Klanderma, and Khoury (2001) shows that tables help learners to identify the pattern in a relationship. I also observed his being good at pattern finding during the class observations. When they were playing the "guess my rule" game, John was actively involved in the activity. He was not usually that active in the class. When the class was working on the worksheet on finding the rule from the table he was also quite successful and fast.

The 2nd problem in the 1st interview required him to find a rule for the given verbal expression. He made a table and easily found the rule ($y=6+x$). For Lisa's jogging card (card-12 in Appendix C) he could write $y=3$. His strategy to find the pattern from a table was basically trial and error. However, when there was a context related to the problem he found the equation easier by also considering the context. For example, he easily found the

equation for Adam's savings, $y=200-5x$ (problem-3 in the 4th interview).

To translate graphs to equations as in the 1st interview 3rd problem, he was making tables and then finding the rules. To sum it up, when he was asked to find a rule from any kind of representation, he made a table and found the rule from it.

Classification

John's low achievement affected the way he classified the linear equations. The third interview included questions about classifications. He was supposed to sort the cards into piles according to a criterion. At the beginning he could not find any way to make the grouping. He did not have enough knowledge to analyze the relationships in depth and then to divide them into groups. Then I composed groups and asked him to guess my criterion. When I grouped by representation, he could find it but when I went by the type of the relationship he could not find it. For the 4th question in the 3rd interview, I asked him what the relationship between card-7 and card-12 was. He could write card-12 in terms of symbols ($y=3$) but interestingly, he said that the relationship between card-7 and card-12 was that both of them were odd numbers.

John was able to classify tables according to increasing, decreasing, and constant as were all the other three students. For graphs, he showed a similar pattern to David. For card-4 in the 3rd interview he said: "I was thinking since like the line is in the middle of the graph, it is going both down and up so this one is neutral" (Interview 3). He classified card-5 in the 3rd interview as a decreasing graph and explained that "the line is on the negative side of the graph and they are both going down.

This one is going down negatively and that one is going up negatively. That makes it decreasing" (Interview 3). However, again similar to David, for the contextual graphs he had a lot more to say. (See his explanation about Moshon's savings in the translations to verbal section above and his interpretations for the problem-1 and problem-2 in the interpretation section below.)

John was comfortable using other representations or different methods to find a solution. When he was asked if an equation was increasing, decreasing or constant, he sometimes looked to see if the equation included a negative number or not. For example he classified $y=2x-1$ as decreasing because of the "take away 1" (Interview 3). However, more often he made a table to find out if an equation was increasing, decreasing or constant. In the 4th interview, we had the following conversation:

I: When you see an equation, can you tell if it is increasing or decreasing? For example consider $y=2x-10$. Can you see it right away or do you need to work it out?

J: Work it out.

I: So, is this increasing or decreasing?

J: I think it is increasing.

I: Why?

J: 2 times 5 is 10, take away 10 is zero, 2 times 6 is 12, take away 10 is 2, so it keeps going up.

I: So, you are making a table in your mind?

J: Yes. (4th interview)

He used a similar process when he explained $y=-5x$ was decreasing as discussed in the translations to verbal representation section above.

Although his method for finding the slope from a graph was wrong, he was thinking it was correct. He was picking a point from a graph and dividing y by x to find the slope. When I asked him to find the slope for an equation ($y=2x-1$), he first made a table and then sketched the graph of the equation and found the slope from the graph. Unlike David, he was flexible in using different representations to reach the solutions.

For verbally represented linear relationships, John did not have difficulty in classifying them once he understood them. For Lisa's jogging function (3rd interview, card-12) he said it stays the same, for Phil (3rd interview, card-8) he said his savings increases, and for Deanne (3rd interview, card-3), he said: "A decreasing relationship because she is spending 5 dollars every day and it is decreasing from her paycheck" (Interview 3).

Interpretation

Being a high spatial ability student, John was able to develop strategies to solve problems. For example, for Moshon's savings (represented as a graph, see Appendix B) he reasoned as follows to find Moshon's savings for the 40th week:

Because of 10 weeks, I am guessing there are 3 more 10 weeks. If you add 3 more sets of 10 weeks that would be 40. I was thinking if 10 equal 60 dollars in 10 weeks. I just did 60 times 4, which gives you 240 dollars what you get in 40 weeks. I mean there are 3 more sets of 10 weeks times sixty dollars, so four times sixty. You should join them of course. (Interview 2)

He was not considering the fact that Moshon's money was 10 dollars at the beginning of the year (if he was high

achieving, he would probably consider it) but he was able to develop a proportional reasoning approach for his solution.

For problem-3 in the 4th interview, different from David (low achieving-low spatial ability) and Lara (high achieving-low spatial ability) and similar to Rose (high achieving-high spatial ability), he said Linda's savings will probably catch up to Adam's savings. Moreover, he found the time of intersection, the 20th week. We had the following conversation:

I: How much money will she have then?

J: 100.

I: How did you find it?

J: Since this one is 10 weeks and this is 200, if you do 10 more weeks, that will be 1 more 100 and she will have 100 left.

I: How did you find Adam's money?

J: That was like... they are 50 dollars apart.

Since they are 50 dollars apart... I just compared their weeks. I can not remember what I did. It skipped in my mind. (Interview 4)

He tried to remember his strategy but he could not. He somehow compared two tables, considered the difference between them and found Adam's money. When I insisted that he give an explanation, he used proportional reasoning based on the first 10 weeks of Adam's table and found Adam's money for the 20th week. From my own experience, I believe that proportional reasoning may require visualization. Especially in this last example to compare the two tables, he might have needed to keep in his mind the numbers in two tables, the difference between the two

savings, and then visualize the further weeks by reasoning about the ratio.

For comparison, he was able to compare two tables such as "this table is increasing by 2, this table is increasing by 3, and so this table is increasing faster" (Interview 4). He compared the two parallel lines representing the temperature in two cities by saying: "First one will be hotter and this one will be colder." He could also state that they were increasing at the same rate. For the baby growth problem in the 4th interview, he said that baby-A was 2 pounds and baby-B was 4 pounds when they were born and baby-A was growing faster than baby-B. He explained this as follows: "Because, for the 6 month, when baby-A was 8 pounds and baby-B was 7 pounds" (Interview 4). Instead of observing the general growth he compared their weight at a specific time. This might be due to his extensive use of tables because on a table John was analyzing the relationship point by point. Another reason might be he was not making any connection between the slope and the rate of change. Although he says slope is rise over run, his work shows that he did not really understand the meaning of slope. He was able to draw the points when the slope was given but to find the slope on a graph, he just picked a point and divided y by x . (He did this when discussing the slope of card-5 in interview-3, and problem-2 in interview-4.) This is an effect of his low achievement on his use of multiple representations. If he understood the concept of slope, he would make better comparisons between different linear relationships as evidenced in the high achieving-high spatial ability student.

An interesting finding about John was revealed when I asked him to interpret $y=5$ in the 4th interview. He made a

table with all y values 5 and he said that it was constant. Then, I asked him to imagine its graph. He said "it will be increasing, going up" (Interview 4). This suggested that he could not make a conceptual connection between the table and the graph.

John compared Adam's savings (table) and Linda's savings (equation) by translating the equation into a table. In the 2nd interview, for the last question he compared the savings of Dina (table) and Moshon (graph) as follows:

I would say Dina; I guess she is increasing by a little bit more money than Moshon is. Because at the 8th week she has 56 dollars and Moshon has 50 dollars and the 40th week Dina has 280 dollars, and Moshon has 240 dollars. (Interview 2)

Once again (similar to his interpretation about the baby growth problem in the 4th interview), instead of comparing the weekly increase, he picked two points and compared them.

Preference and Tendency

For part-c of the 2nd problem in the 2nd interview he picked Dina (table) and Moshon (graph) to make the comparison. For question 7 in the 3rd interview he picked a table. His choices revealed that his preferences may be tables and secondly graphs. Obviously, it can be observed from his work that his tendency was using tables for the problem solving process. Many times he used tables as a tool to reach the solutions.

Even though he was a high spatial ability student, he usually did not tend to use graphs during the problem solving process. I believe that this was due to his low achievement level. He did not have a comprehensive

understanding of graphs. Nevertheless, he could develop strategies on graphs (e.g., Moshon's savings) and he plotted the graph of an equation to find the slope. Thus he was not totally ignoring the graphs. I believe that because of his achievement level he tended to use tables a lot since tables are easier to analyze (Kieran, 1992). He was good at pattern finding, which may be another reason for his extensive use of tables since it is easy to see the pattern from a table (Hines, Klanderma, and Khoury, 2001).

The Case of Lara (High achieving-low spatial ability)
Translations

Translations to graphs. Similar to John, Lara also used tables for translations between different representations. For example, for the 1st question in the 2nd interview she made a table to go from the equation to the graph. She also used positive numbers only. Unlike John, Lara connected the points to make a line.

In the 1st interview for the first problem (part-b), Lara confused the x and y axes while she was plotting the points on the coordinate axes. For example, instead of (1, 15) she marked (15, 1) but she wrote the rule as $y=15x$. Although she graphed the points along a line with a very small slope and created an equation with a large slope, this contradiction did not bother her.

Translations to tables. Lara did not have any difficulty in translating other representations to tables. Moreover, she often used tables for translations from verbal statements to rules, graphs to rules, and symbolic representations to graphs such as in the second and third problems in the 1st interview, and 1st problem in the second interview. In the 4th interview, for the 3rd problem she

constructed a table for Linda's savings ($y=300-10x$) to compare it with Adam's savings (table).

Additionally, she used tables to determine slopes. For example, in the fourth interview when I asked her to find the slope from an equation, she made a table and found the slope correctly.

Translations to verbal statements. Lara's translations from tables to verbal statements were clear and detailed. She explained Dina's savings (table, 2nd problem in the 2nd interview) as "Dina, her savings was increasing by seven weekly, and by the 40th she had 280" (Interview 2). I showed her two tables in the 4th interview and asked her to find the table that increased faster. She selected one and correctly explained as follows: "Because this one is increasing by 3, and that one is increasing by 2" (Interview 4).

She explained Moshon's savings (graph) as "Moshon, his savings increased on the 8th to 40th week by 160 dollars" (Interview 2). Later when I asked for the weekly increase she added that he increases his money 5 dollars each week. In the 2nd interview, for the first question she explained how she plotted a point as

X is 1 and y is -3. You go over 1 because x means run and y means rise and this case you go down because it is a negative and positives are right and up and negatives are left and down. So you run 1 and then you rise over, fall, I mean you fall -3. (Interview 2)

The effect of her high achievement could be observed in the previous explanation (connecting rise with y and run with x).

In the 1st interview, she related a pair to the rule by saying, "The x is 3 and the y is 45. Because when you multiply 15 times 3 you get the y, which was the 45." For Danny's savings she said, "And Danny his savings decreased; by the 40th week he had no money" (Interview 2). When I asked her what $y=5$ meant, she said, "When you have a table, the number on the y equals 5, all of them" (Interview 4).

Translations to equations. This type of representation was the hardest type for Lara (actually for all of them). She was not very good at pattern finding, which made it harder for her. When the class was playing the "guess my rule" game, she did not join the game (as David). The next day, she completed the exercise sheet about rule finding from tables but was slower than John and Rose.

To find an equation from a graph or a verbal representation, the majority of the time she made a table. For easy rules, she could find them (such as the second problem in the 1st interview, $y=x+6$). For more complex rules she had difficulty. For instance, for Moshon's savings (graph) in the 2nd interview, she made a table and tried to figure out the rule but she could not see the pattern. Rose (high achieving-high spatial ability) also had a hard time figuring out the pattern. Even David (low achieving-low spatial ability) could see the pattern easier than the two of them. I compared the three students' processes. Then I recognized that David reasoned about the context at the beginning of the solution and found the rule easily. Rose and Lara concentrated on the table and the numbers without relating it to the context. Only after I suggested that they think about the savings pattern could they find the rule. Another example for relating the rule to the context

is from the 3rd interview. Lara and I had the following conversation about Deanne's savings (represented verbally):

I: Can you find an equation for Deanne's savings?

L: (She works on it.) I know it is $5 \cdot x$ but I do not know where to put it. Is it division?

I: What is her money after 2 weeks?

L: 350, from her paycheck?

I: Did you understand the problem here?

L: No because I do not know how I am supposed to write an equation.

I: She earns 150 dollars, and then she spends 5 dollars every day for her lunch. For example, how much money will she have after 2 days?

L: How much will she have spent?

I: No, not will have spent, will she have?

L: 140?

I: Yes, after 3 days?

L: 135.

I: Yes, can you find a rule for this relation?

L: Each day you decrease by 5 from the original number.

I: Yes. Can you write it in terms of x and y ?

L: (She writes it correctly).

I: Very good. How much is it decreasing by each day?

L: 5. (Interview 3)

Obviously, if she relates the rule finding to the context, she figures out the rule much easier. For Lisa's jogging function ($y=3$) (card-12, 3rd interview) and for Phil's savings ($y=6x$) (card-8, 3rd interview) she could write the rules without making a table.

Classification

Similar to the other three students, Lara could classify tables according to tables being increasing, decreasing, and constant (e.g. Dina's savings in the 2nd problem of the second interview, and the table comparisons in 4th interview (see translations to table section above)).

In the 3rd interview, for the 1st and 2nd questions she classified card-4 as increasing and card-5 as decreasing graphs. However, for the last question, she classified both of them as increasing. She explained card-4 is increasing by 2 "because when you go to 1 it is 2, when you go to 2, it is 4" (Interview 3). She was reading the graph as reading a table. In the 4th interview I asked her about card-5 again. First, she said it was increasing and then she said it was decreasing and explained, "Because of the negatives I guess. I do not know. It just looks like it is decreasing." For contextual problems including graphs, she classified them correctly. In the 4th interview for two parallel lines representing the temperature in two cities, she said both are increasing and then we had the following conversation:

I: Can you compare if they are increasing the same or one is increasing faster than the other?

L: They are increasing the same.

I: Why?

L: Because they are both increasing by 1.

I: How did you find 1?

L: Because this; when it is on 1, it goes up to 2, when it is on 2, it goes up to 3, 3, 4, 4, 5. This one goes negative 2 to negative 1, 0 to 1, 1 to 2 and so on.

I: Good. But the temperature in two cities is not exactly the same. What is the difference between them?

L: This temperature started off at 1 and increased, this temperature started at negative two and increased. (Interview 4)

For verbal representations, Lara could classify them according to their being increasing, decreasing, or constant. She classified card-3 as decreasing, card-8 as increasing, and card-12 as constant in the 3rd interview. For Yonni's savings (verbal, constant) she said, "Yonni stayed, I mean, kept his 300 dollars throughout all the weeks so he did not gain or lose anything" (Interview 2).

For equations, sometimes she just looked at the equation and analyzed it according to the addition or subtraction operation. For example, she classified card-2 ($y=2x-1$) as a decreasing equation. Sometimes, she made a table to understand if an equation was decreasing or increasing especially after she made an incorrect interpretation about an equation by considering the subtraction operation. For example, she made a table to explain that $y=-5*x$ is decreasing and she explained $y=3*x+1$ is increasing as follows "Because when you fill in 1, it will give you 4. And then when you fill in 2, it is 7. So it is increasing"(Interview 4). She classified the constant equations as constant relationships. For example, she put card-7 with the constant equations for the last question in the 3rd interview.

Her understanding of slope was definitely better than the low achieving-low spatial ability student and the low-achieving-high spatial ability student but not as good as the high achieving-high spatial ability student. Lara could

find the slope from a table and also from a graph. (Very rarely did she have difficulty in finding the slope from the graph but she reasoned about the rise over run not y over x as John and David had done.) The majority of the time Lara did not relate a relationship's increasing, decreasing or constant nature with its slope.

Even though she was a high achieving student, she could say an equation was constant and its graph was increasing:

I: Is $y=5$ increasing, decreasing, or constant?

L: Constant.

I: Can you imagine its graph?

L: It will be going up, increasing.

I: Can you find the slope of this one?

L: $0/1$. (Found it from the table) (Interview 4)

The connections between different representations were not firmly established for her. It may also be suggested that she could not visualize the graph of $y=5$.

Interpretation

In the 4th interview, when I asked Lara to compare two tables she was able to find the one that was increasing faster and provided the following explanation: "Because this one is increasing by 3, and that one is increasing by 2" (Interview 4).

In the second interview, she picked Danny's (equation) and Yonni's (verbal) savings for comparison. She said, Yonni started off with more money than Danny and by the 40th week he still has more than Danny. Danny's decreases because he is probably spending his money. And Yonni kept his money at a constant rate. He did not spend money, he did not lose money, and he did not gain any. And at the end,

40th week, Yonni had the most money of all.

(Interview 2)

Then I asked her to compare Yonni's savings with Moshon's savings (graph). She compared them as follows:

Moshon, his savings increases by weekly by 5 dollars. And Yonni he still remains the same so he did not gain any money but Moshon gained money. And at the end, because Yonni kept his 300 dollars he did not spend it; he had more than Moshon because Moshon's money increased at 5 dollars each week. (Interview 2)

In the 3rd interview, for the first question she first grouped the cards (card-1, card-2, and card-10) by representation. Then, when I asked her to find a second way she did it by considering if the cards had negative or positive numbers. She grouped card-2 with card-10 since the equation in the card-2 had a subtraction and the table in the card-10 had a negative y value. Then I guided her to make a table for the card-2 and then she explained, "these two are increasing by 2, this one is decreasing by 3" (Interview 3). During the 3rd interview she usually tried to match the cards by obtaining one of them from another card instead of seeing the relationship in general. Similar to John, she also had a point-wise focus on representations. At the end of the 3rd interview I asked her to compare card-1 (table), card-2 (equation) and card-8 (verbal) in terms of which one was increasing fastest. She selected card-8 and correctly explained that it was increasing more than the others.

One of the interesting comparisons she made is from the 4th interview. Although she compared the two parallel

lines in problem-1 very well, for problem-2 we had the following conversation:

L: They are both growing at the same rate but baby-A started off with 2 pounds and baby-B started off at four pounds. When they get to the 4th month they are at the same pounds. Then, baby-B started increasing by... Baby-A is...Baby-A reached 9 pounds. I do not know. They are increasing the same amount.

I: So they are increasing same?

L: Yes.

I: When you say they are increasing at the same rate, does that mean their slope will be equal?

L: No.

I: What is the slope of the baby-A?

L: 4/1

I: How did you find it? Which point are you using?

L: This right here. No, 2/4.

I: What about baby-B?

L: Wait, baby-A is 4/4, baby-B is 2/4.

I: Which one can I say? Baby-A is growing faster than baby-B. Baby-A is growing faster than baby-B. They are growing same.

L: Baby-B is growing faster. No, baby-A is growing faster.

I: Why?

L: Because baby-A rises bigger than baby-B.

I: If baby-A is growing faster can we say they are growing at the same rate?

L: No. (Interview 4)

It was probably due to her low spatial ability level that at first she thought the two babies were growing at the same rate. However, thinking about slope helped her to interpret their growth.

I would like to mention another example where I think her spatial ability level had an effect on her response. For the 4th problem in the 1st interview, to draw a steeper line than the line she had drawn, she first decided on a slope and then drew it. The two lines were almost looking parallel. Rose (high achieving-high spatial ability) responded to the same question by directly drawing a line that looked much steeper than the one she had originally drawn.

In the 2nd interview, for the 1st question I asked her to find the y value for $x=0.5$ and show it on the graph. She miscalculated and found a point which the line did not pass through. I asked her if that was okay, and she said yes and added that sometimes there were points in the middle like that point. Rose did the same kind of error but that bothered her, she checked her calculations and found the correct point. Another observation about Lara was that she hardly ever checked her answers.

In the 4th interview I presented two equations representing temperature in two cities and asked her to compare them. (This question was asked to Lara and Rose only. The equations were $y=3*x-2$ and $y=4*x-5$.) Lara translated the equations into tables and compared them: "This one is increasing by 3, and this one is increasing by 4; this one is increasing the fastest" (Interview 4). So she was using tables to make sense of the equations but she did not necessarily connect the table with the context as in finding the rule for Moshon's savings (2nd Interview).

Preference and Tendency

In the 2nd interview, she selected Danny's savings (equation) and Yonni's savings (verbal) for comparison (she said they were easy to compare for her). In the 3rd interview, she selected a table (card-1) for question 6. I asked each student if they preferred to use mental math or rules during the problem solving process in the 1st interview for the 1st problem. Lara said she preferred equations and she said that equations work best for her. Connecting this with her selections, I would say her preferences of representations were equations and tables.

During the problem solving process, I observed that she had a tendency to use tables the majority of the time. For the second problem in the 1st interview, even though she found the equation for the baby girl's growth from the table, when I asked her the baby's weight for the 11th month, she made the table longer and found it. Also, in most of the other problems she made tables to make sense of other representations including equations.

The Case of Rose (High achieving-high spatial ability) Translations

Translations to graphs. Different from the other three students, Rose plotted the points for both positive and negative x-values for the first question in the 1st interview. Nevertheless, the context was not proper for negative x-values for that question. In the 2nd interview, for the first question, she plotted the line again for both negative and positive values of the x axis. Moreover, she was the only student who did not make a table for that question. She found the slope from the equation and directly drew the line by reasoning about the slope. However, for some other questions she first made a table to

plot the line of an equation such as in 4th interview, $y=3x-2$ and $y=4x-5$. She was able to translate tables, verbal statements and equations into graphs either by using a table, by considering the slope, or by just plugging numbers in her mind.

She was able to visually draw lines, too. For example, for card-9 (table) in the 3rd interview, while comparing it with a constant graph she said that they were both lines and she drew a horizontal line with her hand. I would like to remind the reader that the other students compared them by saying that they were staying the same (or similar statements). I will give further examples in the proceeding sections about her visually drawn graphs.

Translations to tables. Rose was able to translate any kind of representation into tables, easily. Her work for the 4th question in the first interview and problem-5 in the 4th interview can be given as examples. Moreover, she sometimes used tables to make sense of other representations or to figure out the rules. For instance, in the 1st interview for the 2nd problem, she made a table to figure out the rule for the baby girl's growth.

Translations to verbal statements. She effectively explained tables in words. For Dina's savings (table) in the 2nd interview she said, "For Dina, she increased throughout the year weekly by seven dollars."

She was also able to explain graphs well. She interpreted Moshon's savings (graph) as follows: "Moshon, she increased five dollars weekly throughout the entire year" (Interview 2). In the 4th interview, for the two parallel lines we had the following conversation:

R: They are increasing the same. Because the slope is the same, it is only 1/1.

I: For both of them?

R: Yes.

I: But there is a difference between them.

R: One is of course colder than the other or the actual starting points are lower or higher than one another. (Interview 4)

The majority of the time, when the question involved a graph, Rose found the slope first (e.g., 1st interview 3rd question: she found the slope as negative 1 over 1 as soon as she read the question). For Yonni's savings (verbal), she said, "Yonni never changed and stayed throughout the whole entire year 300 dollars" (Interview 2). Then I asked her to guess the slope for Yonni's savings. She said "zero over 300."

For Danny's savings (equation) she said, "Danny, he actually lost money weekly" (Interview 2). First she said he lost money by 200 but then corrected herself and said it was by 5 dollars. In the 1st interview she related the pair (2, 30) to the rule by replacing 2 for x and 30 for y; she said: "like replace y and x with those. It would be 30 equals 2 times 15" (Interview 1).

Translations to equations. Similar to the other three students, this type of translation was also the most difficult kind for Rose. However, she was much better at pattern finding than David and Lara. In the mathematics class, she was actively involved in the "guess my rule" game. The next day, she did well on the worksheet that was about rule finding from tables (Appendix E1). She was fast and successful. This class took place after the third interview. Until that time no student reasoned about the relationship between the coefficient of x and the increase in x values. In that class, I asked Rose, Lara, and John if

they recognized that pattern while they were finding the rules. It was only Rose, who realized that relationship.

Rose's problem solving process revealed differences from that of the other three students. First of all, in the 1st interview for the second and third problems (Appendix A), after she read each problem she tried to find the pattern without making a table. In the third interview, she drew the line for $y=2x-1$ without making any table. She tried to make translations between different representations without using a third representation. This does not mean she did not use other representations in the problem solving process. Whenever she felt she needed a third representation she used it. For instance, while solving the second problem in the 1st interview (Appendix A), she found some rules, checked them by plugging in numbers, understood they were wrong and then made a table to find the correct rule. After making the table, she found the rule easily. Another difference in her problem solving process from the other three students was her checking of answers. She usually did the checking process in her mind. She was very fast in her responses, which caused her to make many mistakes, but the majority of the time she corrected herself.

For the third problem in the 1st interview (Appendix A), she did not make any table; she had difficulty but found the rule. For this problem, her thinking was very similar to that of David. She thought the rule involved subtracting something. But since she was very flexible in changing her strategy and she was checking her responses, she found the rule easier than David did.

In the second interview, she struggled to find the equation for Moshon's savings (graph) (Appendix B question

2). After she made a table, she still had difficulty in finding the rule. Earlier I discussed a similar difficulty in the case of Lara. I believe that both had difficulty because they concentrated on the numbers in the table. They did not relate the numbers to the context. When I suggested to Rose to think about the context (the weekly increase in Moshon's money), she found the rule easily like Lara did. Similarly, in the 3rd interview Rose tried to write the equation for card-3 (verbal), she had difficulty. However, when I lead her to consider the context she found the equation:

I: What is her money after two days?

R: it is going to be 140.

I: After 15 days?

R: 150 minus 15 times 5.

I: So what is the equation?

R: It is $150-5x$. (Interview 3)

Rose found the rule $y=3$ for Lisa's jogging easily (card-12, 3rd interview). This may be due to her ability to visualize constant values and her strong understanding of constant functions. In the 2nd interview, while explaining Yonni's savings (verbal, constant), she made a horizontal line with her hand and also found the slope zero for Yonni's savings.

In the fourth interview, for Adam's savings (problem-3, Appendix D) she easily found the rule to determine his savings for the 15th week. There may be two reasons for this. Firstly, she was already thinking about the context so that connection might have helped her find the rule. Secondly, she was learning during the interviews. Also before this interview, the students had learned about linear functions in their class and she improved her

understanding about linear equations. By this time, it was easier for her to figure out a rule from a table.

Classification

Rose was the only student who connected the slope with the linear relationships' increasing, decreasing, or constant nature. For the 1st question in the 2nd interview (Appendix A), she found the slope as negative 4 over 1 and said that it was decreasing. However, she made a mistake and drew an increasing line. Then she recognized that contradiction and corrected her mistake. She could always find the slope from a graph and tell if the graph was increasing, decreasing, or constant. For problem-3 in the 1st interview, she said it was a decreasing relationship and, for card-5 in the 3rd interview (Appendix C) she found the slope correctly and said that the graph was decreasing.

She was able to classify tables as increasing, decreasing or constant. In the third interview, for the last question she classified all tables correctly. Moreover, in the fourth interview, when I asked her to compare two tables, she used the slope of the two tables to find which one was increasing fastest. However, in the third interview, when I asked her to find the slope for card-1 (table), she found it by dividing y by x. Then I told her that the slope for that table was 2/1 and explained it. Later in the same interview I asked her for the slope of the same table and she found it as 2/1. In the 4th interview, I asked her to find the slope for a table. She found it correctly and explained the solution as "there are 3 differences with the y and only 1 difference there (x)." She also found the slope correctly for Adam's savings (Problem-3 in the 4th interview). Therefore It can

be concluded that she was able to find the slope from a table for the majority of the time.

She explained card-10 (table) (Appendix C) was decreasing: "This is decreasing. You are at zero it is 10, then you are at 1 it is 7 so you are decreasing. (She shows it as a graph by motioning with her hand while talking.)" (Interview 3). This is an example of how she was mentally drawing the graph of the table.

Rose did not have any difficulty in classifying verbal representations. She classified Yonni's savings (2nd interview) as constant, card-3 (3rd interview) as decreasing, and card-8 (3rd interview) as increasing (she also converted card-8 to an equation, $y=6x$ and found the slope as $6/1$).

The only problem with classifications that Rose had was with the classification of equations. I will exemplify this with the following conversation from the 4th interview:

I: $Y=2x-5$, is it an increasing or decreasing relationship?

R: Decreasing.

I: Why?

R: Because you are subtracting from whatever x comes out after being multiplied by 2.

I: Is it a negative slope or positive slope?

R: Negative.

I: Can you explain more why it is decreasing by using numbers or other things?

R: The only way I can probably explain more is if I do a graph.

I: Can you do that?

R: Say x equals 1 that will be negative 3, say 0, it is negative 5, and actually I was wrong. It increases. (Interview 4)

When she translated an equation to a graph she did not have difficulty in classifying that equation because her understanding of graphs was strong. In the 3rd interview, I asked her to separate card-2 ($y=2x-1$), card-5 (decreasing graph), and card-11 ($y=-5x$) into two groups. She drew a graph of $y=2x-1$ on the paper and she said she mentally drew the graph of $y=-5x$. Therefore, she grouped card-5 and card-11 together because of "negative slope" and card-2 alone. For card-2 she said "this one increases, positive" (Interview 3).

In the 4th interview, she classified Linda's savings ($y=300-10x$) correctly. We had the following dialog:

I: What about this one?

R: I am thinking it is decreasing by 10.

I: Can you prove it?

R: I would prove it by a graph but last time it did not work out.

I: But last time you guessed wrong.

R: $300-10x$ is decreasing because no matter what x is going to be it only will become larger and the larger the number you minus from 300 the less outcome it will be. (Interview 4)

Then I asked her to find the slope for Linda's equation. Until that time, she did not know how to find the slope from an equation. She said that they had learned it in the class but she forgot how to do it. What she said for Linda's savings was interesting:

R: The slope here is negative 10/1.

I: Why?

R: Because you are subtracting 10 and 10 is connected with x . That is what I figured out. I finally understood getting the slope here; the slope is just whatever x is to. That took me a long time. I just finally solved the pattern.

(Interview 4)

This showed that she was learning during the interviews.

Interpretation

Rose compared tables well and except for the example given in the classification part, she was able to find the slope from a table and connect it to the type of the relationship. In the 4th interview, I showed her two tables and asked her to find the fastest increasing one. She found the fastest increasing table and explained, "This one because of the slope" (Interview 4). She found the slopes for the two tables correctly.

In the 2nd interview, she selected Moshon's savings (graph) and Danny's savings (equation) for comparison. This selection reflected her high achievement since she chose them by analyzing the relationships. She said that "the easiest are Moshon and Danny because basically their savings and decreases I guess were opposite. They were the same slope just negative and positive" (Interview 2).

To interpret equations, she usually translated them into graphs. (See the example from the 3rd interview about cards in the classification section above.) In the 4th interview, I asked her to compare the temperature in two cities where the temperatures were given by $y=3x-1$, and $y=4x-5$. She said, "I will do a graph as I usually do." Then she made tables for both of them and graphed them. She compared the temperature in both cities as follows: "Basically I see the only difference is that one city

started off colder than the other and it increases more largely with temperature than this one does because the rise and run are different" (Interview 4). As usual, her interpretations about increase involved slope. This showed that her understanding of slope was good and she made connections between sub-concepts within linear equations (slope and classification of the relationship).

Another example about her understanding of graphs and the connections she was making between equations and graphs was from the 2nd interview, 1st question. Similar to Lara, Rose also made a mistake and plotted the point (0.5, -1) incorrectly so that it was not on the line. (Remember this did not bother Lara.) For Rose, something was wrong and she said, "It is not on the line and they should fit especially if x is a variable" (Interview 2). Then I suggested she check her steps for plotting the point and she corrected her mistake.

Her comparison of the two babies' growth in the 4th interview was interesting since it showed that she did not feel that she needed to connect the concept to the context. Here is the conversation we had:

R: I do not know how exactly how to compare but I know that the slopes are different.

I: Which one is growing faster?

R: As in which way you going through by x or y because A would be growing faster than B in y but then B is faster than A on x.

I: Let me remind you of something. Y axis shows the pounds and x axis shows the time.

R: Okay.

I: Now can you tell whose pound gets bigger faster?

R: A. (Interview 4)

As the above examples show, her interpretations reflect the effect of her high achievement and high spatial ability level. Her achievement affected the way she connected the representations, the concepts, and analysis and understanding of each representation. Her spatial ability affected her ability to visualize each representation and her preference and tendency toward representations that is discussed in the next section.

Preference and Tendency

For the 2nd problem in the 2nd interview, Rose selected Moshon's (graph) and Danny's (equation) savings for comparison. For the 6th problem in the 3rd interview, she selected card-8 (verbal) for analysis. These selections reflect her achievement level. For Moshon and Danny, she reasoned about how each relationship changes. (They were changing by the same amount, but one was increasing and the other was decreasing.) For card-8, she said that it was the easiest for her. Actually it was just $y=6x$ (when converted) that is a fairly easy linear relationship. Therefore, I would say that her preference of representation was whichever one was the easiest for her to analyze.

Her work revealed that her tendency was for using graphs. In the above sections, there were many examples in which she translated other representations into graphs to interpret them, to classify them, or to find the slope. One more example would be from the 3rd interview. For the 5th question, she was supposed to match card-8 to another card. Although the given information matches an equation (increasing) to another equation (decreasing), she matched card-8 (verbal, increasing) to a graph (decreasing). She could have matched it to a decreasing verbal question or

even another decreasing representation but she selected a graph. Additionally, when I asked her about her choice of Moshon's and Danny's savings to analyze, she said that "I do not particularly like graphs but it is easiest to see visually."

CHAPTER 5

Discussion

In chapter 4, I presented the findings from interviews and classroom observations for David, John, Lara, and Rose with respect to translations, classification, interpretation, and preference and tendency towards representations of linear equations. I described how each student used and understood each representation (verbal, tables, graphs, equations), translated between multiple representations, interpreted different representations, and preferred and tended to use each representation.

The purpose of this research study was to examine the effects of achievement and spatial visualization on students' use of multiple representations. In this chapter, I will give answers to the research questions.

How does spatial visualization affect students' use of multiple representations?

I would like to remind the reader that among the four students David and Lara have the low spatial visualization levels and John and Rose have the high spatial visualization levels.

In this section I will present examples in order to compare students whose achievement levels are the same but spatial ability levels are different. First, I will compare David and John who are both low achieving but at different spatial ability levels. Then, I will compare Lara and Rose who are both high achieving but at different spatial ability levels.

When the translation sections of the findings in Chapter 4 were analyzed for David and John, it was seen that they show a similar ability in translations to tables and graphs. For the 4th problem in the 1st interview, they

both drew the line with the given slope as points without connecting them. Moreover, they both preferred to operate with positive numbers and for graphs they both plotted points on the positive side of the x axis.

For transitions to verbal representations, David was slightly more successful than John, which I attribute to John's low achievement in reading (FCAT level-2 and National Percentile Rank of 28). David was also at level 2 on FCAT but his National Percentile Rank was 54.

They showed a difference in their translations to equations. In the interviews, for any question requiring finding an equation, John made a table. Moreover, he was very successful in finding patterns. On the other hand, David did not tend to use tables or any other representations when responding to questions requiring translations to equations as often as John. This does not mean tables do not help David. For example, for the 2nd problem in the 1st interview, he tried to find the rule directly from the verbal statements. When he could not succeed and stopped trying, I suggested he make a table. He did and found the rule easily. For the 3rd question in the 1st interview, again I suggested he make a table to find the rule from the graph after he had stopped trying to go directly from the graph to the equation. For this question he had difficulty in finding the rule from the table probably due to the difficulty of the rule $y=-x$. In addition to his low tendency to use other representations for translations, David was also not good at pattern finding. This can be observed from the classroom observations and interviews (see the findings for David). For example he could only find the rules $y=x+6$, $y=-2x$, and $y=9x$ from the worksheet (Appendix E1).

Both students were able to classify tables and verbal representations. Additionally, they both showed a lack of understanding of graphs and equations while they were classifying them. The main difference was that John often made a table to classify an equation and did it correctly whereas David just reasoned about negative and positive numbers in the equation to classify it. Moreover, John sketched the graph of an equation to find its slope whereas David just said he did not remember how to find the slope from an equation.

David interpreted contextual problems successfully (see his findings). For example, for problem-2 in the 4th interview he made a global comparison of the growth of the two babies by saying "because baby-A increases by 1 pounds and baby-B increases a half pound in each month" (Interview 4). John also could interpret contextual problems. Since he was usually constructing tables and making translations by using tables, this affected the way he interpreted relationships (Janvier 1978, 1981a, 1981b as cited in Leinhardt, Zaslavsky, & Stein, 1990). John showed a point-wise thinking in his interpretations. From many examples about this that were given in the findings in Chapter 4, one of them was from the 4th interview, problem-2: he explained Baby-A was growing faster than Baby-B by comparing the babies' weight at a specific time instead of observing the general growth.

John was different from David in creating different solution strategies. Whereas David could give up easily, John developed strategies to reach an answer (e.g., his strategy for finding Moshon's savings for the 40th week from the 2nd problem in the 2nd interview).

John's preferences of representations were tables and

secondly graphs and he tended to use tables whereas David's preferences and tendency were for verbal representations and tables and he did not tend to use different representations.

Now, I will compare Lara (high achieving-low spatial ability) with Rose (high achieving-high spatial ability). Both Lara and Rose were able to translate other representations to verbal representations and tables. Their strategies showed differences in their translations to graphs and equations. Typically, for any translation Rose first tried to translate one representation to another representation directly whereas Lara used tables for translations between representations. For example, for question-2 and question-3 in the 1st interview Lara made tables to find out the rules. For both questions, Rose first tried to find out the rule from the original question. For question-3 she was able to find the rule directly from the graph but for question-2 she struggled and made a table. Then, she found the rule. Moreover, Rose was better at pattern finding than Lara was. In the last class that I observed, Rose was the only student among the four who recognized the relationship between the coefficient of x in an equation and the amount of change in x values on the corresponding table. For the 1st problem in the 2nd interview, Rose drew the graph from the equation by reasoning about the slope whereas Lara first made a table and then drew the graph. Another difference between the two students was that Rose drew the lines on both negative and positive sides of the x axis whereas Lara drew the lines on the positive side of the x axis only.

Both Lara and Rose were able to classify tables, verbal statements, and graphs and had difficulties in

classifying equations. When I asked for an explanation for the classification of an equation, Rose translated equations to graphs and Lara translated equations to tables. Similar to this difference in their choice of representations for an explanation, in the 3rd interview, their choices of representations also differed. For questions 4 and 5, Rose related card-2 (equation) to card-5 (graph) and card-8 (verbal) to card-5 (graph). For the same questions, Lara related card-2 (equation) to card-1 (table) and card-8 (verbal) to card-12 (verbal). Actually, I would like to remind the reader that Lara tended to use tables and Rose tended to use graphs for the majority of the time, which is aligned with the following study's finding. In their study, Dreyfus and Eisenberg (1982) found that high mathematics ability students preferred graphical settings and low mathematics ability students preferred tabular settings.

Another difference in their classification was that for increasing and decreasing relationships, Rose always related slope with the type of the relationship. However, Lara just talked about the relationships being increasing or decreasing without mentioning the slope. This may be related to Rose's preference of graphs. Since she was good at graphs and the class learned slope in relation to graphs, Rose's understanding of slope was also good and she connected it with the classification of relationships. Therefore, it may be suggested that spatial visualization has an effect in understanding concepts such as slope.

The difference in Lara's and Rose's preferences and tendency in use of representations can be observed in their interpretation of representations. I asked both students to compare the temperatures in two cities given by equations.

Lara translated both equations to tables and interpreted them successfully. Rose graphed both equations and interpreted them successfully (she made tables to graph the equations).

For the 1st problem in the 2nd interview both Lara and Rose found an incorrect y value for x equals 0.5 and plotted it such that the line did not pass through it. This did not bother Lara and she continued to the next question whereas Rose recognized something was wrong and corrected herself. An example of interpretation of graphs showing differences between the two students is from the 4th interview. For problem-2 where students compared two graphs, Lara said the two babies grew at the same rate whereas Rose compared their growth correctly. Connecting these examples with Rose's ability to interpret slope in relation to different representations and her ability to directly translate one representation to another, I would say that Rose had stronger connections between different representations than Lara did. These differences reveal the effects of spatial visualization. Because of her high spatial ability Rose was able to work effectively with more representations (in particular graphs) than Lara did. Rose was comfortable using tables, graphs, verbal representations, and equations, and easily translated among them whereas Lara was comfortable with using tables, verbal representations, and equations.

A similar difference was also observed in the work of David and John. John (high spatial ability) tended to use more representations than David and he was able to produce more solution strategies than David. Therefore, throughout this study, the two high spatial ability students showed more access to different representations than the two low

spatial ability students. In addition to this, the high spatial ability students were better at pattern finding than the low spatial ability students were. Presmeg (1986) suggested that spatial visualization may help students make generalizations in mathematics through pattern recognition.

How does achievement affect students' use of multiple representations?

I would like to remind the reader that among the four students David and John have the low achievement levels and Lara and Rose have the high achievement levels. In this section, I will present examples in order to compare students whose spatial ability levels are the same but achievement levels are different. First, I will compare David and Lara who have low spatial ability levels but at different achievement levels. Then, I will compare John and Rose who have high spatial ability levels but at different achievement levels.

In terms of translations, Lara was different from David in that she used tables to make translations among the different representations especially to find a rule. David hardly ever made a table to translate between representations. He would easily give up. During the problem solving process, David made errors or gave no answer due to his low achievement level. For example, in the 1st interview, for the 1st question, he had difficulty in writing down the rule in terms of symbols even though he expressed it in words. He also suggested different rules for each point on the graph for the 3rd question in the 1st interview. He did not understand the concept of an equation very well. Lara did not make such errors.

Compared to David, Lara was better at pattern finding. In the class, when they were completing the worksheet on

linear functions (Appendix E1), she found all the rules in section-1 whereas David could only find $y=x+6$, $y=-2x$, and $y=9x$.

David was able to classify tables and verbal expressions but he could not classify equations and graphs. Lara had difficulty in classifying equations but, most of the time she translated equations into tables and then classified or interpreted them. Her use of more than one representation helped her to classify relationships whereas, most of the time David gave incorrect explanations about classifications of equations and graphs due to his low achievement level. Since he did not tend to use other representations, he could not correct his errors.

Lara could find slope from tables and graphs whereas David could not find it from any representation. He was just randomly picking a point and dividing y by x .

David could successfully interpret contextual problems. He even made up a context for an equation to interpret it. Contexts helped him to make sense of representations. On the other hand, compared to David, Lara did not seem to feel that she needed a context to analyze representations. For some problems she forgot the context and concentrated on the numbers. This caused her to have some difficulty in her solutions. However, when I asked questions to help her connect the context with the question she was more successful.

Now, I will compare the two high spatial ability students with different achievement levels. For any kind of translation, John (low achieving) made tables to go from one kind of representation to another. Rose (high achieving) first tried to make a direct translation but she used tables as well. John even used a table to draw a line

whose slope was given whereas Rose directly drew the line from the slope. Actually, John did not show a comprehensive understanding of slope throughout the interviews. (He found the slope from a graph by dividing y by x .) However, being able to use other representations helped John with concepts such as slope, where he had a lack of understanding. By making a table he could plot the line in the above example.

Due to his low achievement, John had difficulty in classifying graphs and equations. However, most of the time he made a table to classify and interpret equations. In contrast, Rose was able to classify graphs. Different from John, Rose drew the graphs of equations in order to classify and interpret them. I interpret this difference as the result of their achievement levels. Even though John was a high spatial ability student, he did not tend to use graphs in the problem solving process due to his lack of understanding of graphs. Tables are easier to analyze (Kieran, 1992); therefore, he preferred tables in his solutions. On the other hand Rose had a comprehensive understanding of graphs. Given this strength along with her high spatial ability, she preferred graphs in her solution methods. She mentioned that she liked to see things visually and they were easier for her.

An example showing differences between students with different achievement levels is from the 2nd interview, 2nd question. After Lara and Rose read the savings of each child, they mentioned that they did not understand Yonni's savings. After my explanation about Yonni's savings they started solving part-a. However, both David and John started solving part-a right after they read the savings of each child. When they came to Yonni, they could not solve the question for his savings. When connected with other

findings for each student, this example suggested that the high achieving students can analyze the problems and their understanding of the problems better than the low achieving students.

Additionally, these findings show that achievement affects students' preferences and use of representations. The two high achieving students in this study had more access to different representations due to their deeper understanding of the concepts.

What are the effects of spatial visualization and achievement on students' use of multiple representations?

I have discussed that both achievement and spatial visualization have effects on students' use of representations. The high achieving and high spatial ability student (Rose) was better able to translate from one representation to another and had stronger connections between multiple representations than the other three students with respect to the concept of linear functions. She had a deeper understanding of linear equations compared to the other three students. The low achieving and low spatial ability student (David) had fewer tendencies to use multiple representations and had weaker connections between multiple representations than the other three students in relation to the concept of linear functions.

The high achieving and low spatial ability student (Lara) demonstrated a deeper understanding of each representation than the low achieving and high spatial ability student (John). However, John was better at pattern finding and creating different solution strategies than Lara. They both tended to use tables.

CHAPTER 6

Conclusions and Suggestions

Conclusions

In this study, the effects of achievement and spatial visualization on students' use of multiple representations have been investigated with respect to the concept of linear functions. Four 8th grade students from different achievement and spatial visualization levels were interviewed. In addition to the interviews, students were observed in the classroom environment to get familiar with the teaching and learning they were engaged in.

Students' work from four interviews and six classroom observations were analyzed in terms of translations between multiple representations, classification of linear functions, interpretation of linear functions represented by multiple representations, and preference and tendency of representations.

It was observed that both achievement and spatial visualization have affects on students' work in terms of the categories above. Given the same spatial visualization levels, the high achieving students were able to access more representations than the low achieving students. Given the same achievement levels, the high spatial ability students were able to access more representations than the low spatial ability students. The high spatial ability students were better at pattern finding and producing different solution methods than the low spatial ability students. The high achieving students showed a deeper understanding of the concept (linear equations) and each representation than the low achieving students.

The high achieving-high spatial ability student showed the deepest understanding of the concept (linear functions)

among the four students. She had the strongest connections between multiple representations. Although she could usually translate any representation to another (she was able to analyze any representation), her tendency was to use graphs.

The high achieving-low spatial ability student showed a deep understanding of the concept (linear functions). She had connections between multiple representations but they were not too strong. During the problem solving process, she had a tendency to use tables to interpret, classify, and compare the other representations.

The low achieving-high spatial ability student showed a lack of understanding of the concept (linear functions). He had a weak connection between multiple representations. However, he was good at the procedure of translating multiple representations to each other and he was able to use different solution strategies due to his high spatial ability. He tended to use tables for interpretation, comparison, and classification of linear relationships represented by different representations.

The low achieving-low spatial ability student had the weakest understanding of the concept of linear equations among the four students. He had weak connections between the multiple representations. The majority of the time he did not tend to use multiple representations during the problem solving process. He was good at interpretation of contextual problems. He preferred verbal representations and tables.

All students could translate different representations to tables. This suggests that translations to tables were the easiest translations for the four students in this study.

All four students could classify tables and verbal representations. This suggests that for the four students in this study the easiest classification was the classification of tables and verbal representations. All students had difficulty in classifying equations, which suggests that classification of equations was the most difficult kind of classification for the four students in this study.

Suggestions

The findings of this research study revealed that students used different representations according to their preferences. When they could not sort out a problem they used different representations in their solution methods. Each student tended to use a different representation according to his/her spatial ability and achievement level. Although the low achieving-low spatial ability student tended to use multiple representations less compared to the other three students, when the researcher suggested he use other representations he could figure out the solution.

Therefore, based on the findings of this research study and other studies, it can be suggested that the use of multiple representations will improve understanding of mathematical concepts (Borba & Confrey, 1993; Yerushalmy, 1997; Brenner, Mayer, Moseley, Brar, Durán, Reed, & Webb, 1997; Swafford & Langrall, 2000; Porzio, 1999; Ozgun-Koca, 1998). By using different representations, the teacher will be able to reach all students who have different needs and preferences. Also, a disadvantage of one kind of representation would be canceled by using another kind of representation that does not have that same disadvantage. Tables could be used as a bridge from arithmetic thinking to algebraic thinking (Friedlander, & Tabach, 2001). It

should be remembered that tables may cause students to think point-wise instead of considering a general interpretation over a function.

Another suggestion that could be drawn from this study is that teaching mathematics in context would help students learn more deeply and make connections among multiple representations. Drawing on context can help all students to better understand a problem and arrive at a solution. This seemed particularly helpful for the low achieving-low spatial ability student.

This study analyzed the effects of achievement and spatial visualization by using case study methodology. The work of the students was analyzed in depth and conclusions were drawn based on the findings. Since multiple representations have gained importance in mathematics education due to their application in teaching and learning mathematics, it would be beneficial to continue research studies in this area. This study was composed of four case studies. To generalize the findings of this study, a quantitative study may be conducted that is supported by qualitative data.

Additionally, it has been observed that the two high spatial ability students were better at pattern finding than the two low spatial ability students. This relationship can be investigated by further studies.

APPENDIX A

Interiew-1 Questions

1. A motorbike travels 15 miles per hour. The table below shows the relationship between the time traveled and the distance.

Time(hours)	1	2	3	4	5	6
Distance(miles)	15	30				

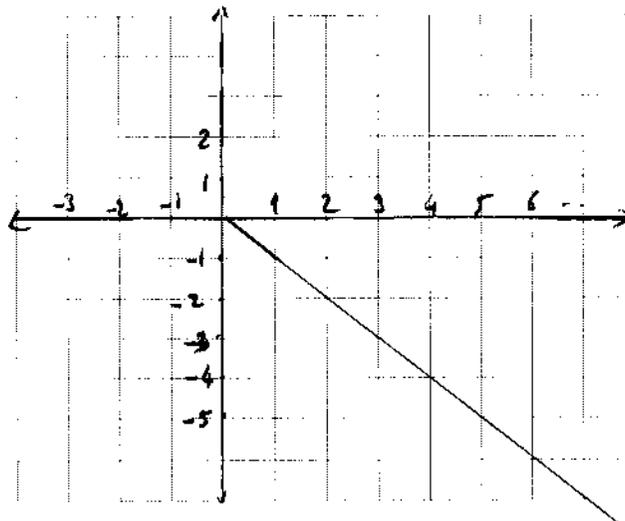
- Complete the table.
- Make a coordinate graph of the data in the table.
- Write a rule that describes the relationship between distance and time.
- Predict the distance traveled in 8 hours and 20 hours.
- Predicted the time needed to travel 150 miles.
- Pick a pair of (time, distance) values from the table. How is the pair related to the graph and the rule?

Modified from Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997.

2. According to a growth and development study of American children, baby girls gain an average of 2 pounds per month during the first 6 months after they are born.

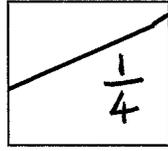
Write a rule that describes the relationship between weight and time (months) for a baby girl weighing 6 pounds at birth.

3. The following graph shows the relationship between temperature and time in North Pole during winter.



Write a rule that shows the relationship between the temperature and the time.

4. In Europe, many hills have signs indicating their steepness, or slope. Here is an example:



This means for each 4 meters
in run the hill rises by 1 meter.

On a coordinate grid, sketch the graph of a hill with the above slope.

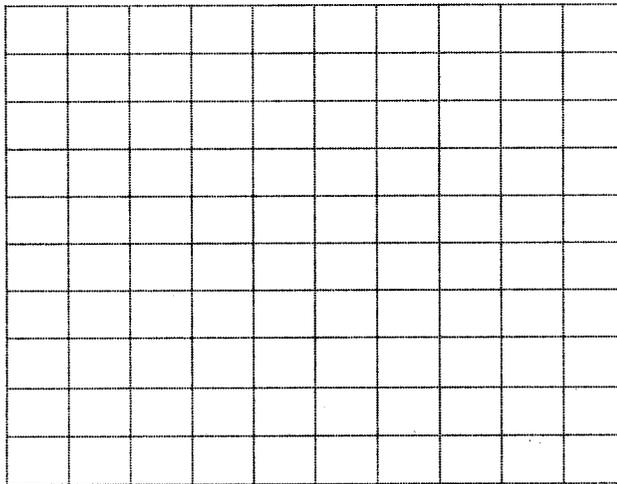
- Generate a table that shows the relationship between the run and the rise with at most four pairs.
- Draw the graph of another hill that is more difficult to climb.

Modified from Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997.

APPENDIX B

Interview-2 Questions

1. Plot the following equation: $y = -4x + 1$.



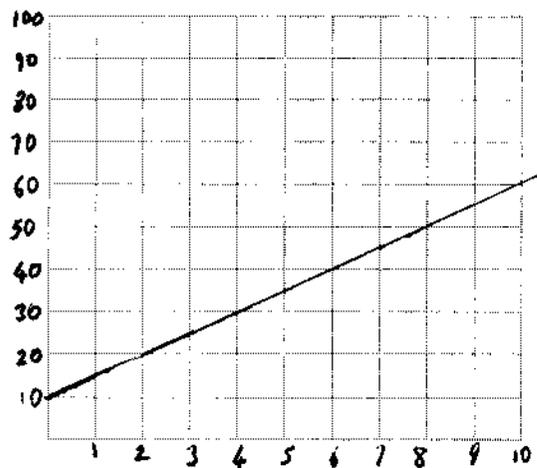
2. The savings of Dina, Yonni, Moshon, and Danny changed during the last year, as described below. The numbers indicate amounts of money (in dollars) at the end of each week.

Dina: The table shows how much money Dina had saved at the end of each week. The table continues in the same way for the rest of the year.

Week	1	2	3	4	5	6	7	8	9
Amount	7	14	21	28	35	42	49	56	63

Yonni: Yonni kept his savings at \$300 throughout the year.

Moshon: The graph describes Moshon's savings at the end of the first 10 weeks. The graph continues in the same way for the rest of the year.



Danny: Danny's savings can be described by the expression $y=200-5x$, where x stands for the number of the weeks and y is his money for that week.

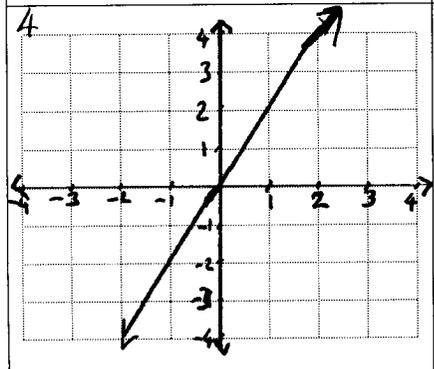
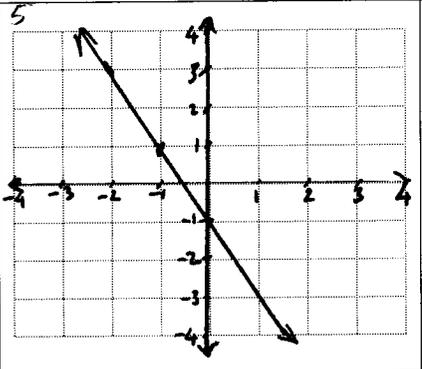
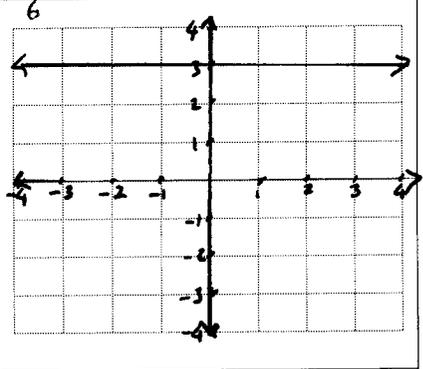
- What is the saving of each child at the end of the 8th week? 40th week?
- Describe in words how the savings of each child changes throughout the year.
- Compare the savings of two out of the four children. Use words like "the saving increase (or decrease)", "who has a larger (or smaller) amount at the beginning (or end) of the year". Use tables, graphs, expressions, and explanations.

Modified from Friedlander, & Tabach, 2001.

APPENDIX C

Interview-3 Questions

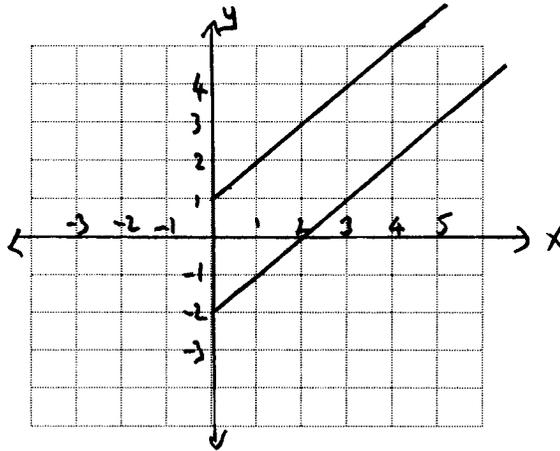
1. Place card 1, card 4, and card 10 into two piles. What criterion did you use?
2. Place card 2, card 5, and card 11 into two piles. What criterion did you use?
3. Dave sorted 4 cards into two piles as follows: card 1 and card 2 in one pile, card 3 and card 11 in one pile. What criterion did he use?
4. Card 7 is to card 12 as card 2 is to what?
5. Card 2 is to card 11 as card 8 is to what?
6. Place card 3, card 6, card 9, and card 10 into two piles. Explain your criterion.
7. Pick one of the following cards: card 1, card 2, card 4, card 8. (Then I asked questions about the picked card such as the type of the function, translation to another representation, evaluation the function at a certain point, etc.)
8. Place all cards into three piles. Explain your criterion.

<p>1</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr> <tr> <td>3</td><td>5</td><td>7</td><td>9</td><td>11</td></tr> </table>	0	1	2	3	4	3	5	7	9	11	<p>2</p> $y = 2x - 1$	<p>3</p> <p>Each month Deanna saves \$150 from her paycheck for lunch. If she spends \$5 per day, how much will she have left after 15 days in Jan?</p>
0	1	2	3	4								
3	5	7	9	11								
<p>4</p> 	<p>5</p> 	<p>6</p> 										
<p>7</p> $y = 5$	<p>8</p> <p>Phil made \$6 per hour of work. How much would he make for 15 hours of work? How much for 35 hours of work?</p>	<p>9</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td>0</td><td>1</td><td>2</td><td>3</td></tr> <tr> <td>-2</td><td>-2</td><td>-2</td><td>-2</td></tr> </table>	0	1	2	3	-2	-2	-2	-2		
0	1	2	3									
-2	-2	-2	-2									
<p>10</p> <table border="1" style="margin-left: auto; margin-right: auto;"> <tr> <td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr> <tr> <td>10</td><td>7</td><td>4</td><td>1</td><td>-2</td></tr> </table>	0	1	2	3	4	10	7	4	1	-2	<p>11</p> $y = -5x$	<p>12</p> <p>Lisa jogs 3 miles everyday. How many miles will she run on the 15th day of the month?</p>
0	1	2	3	4								
10	7	4	1	-2								

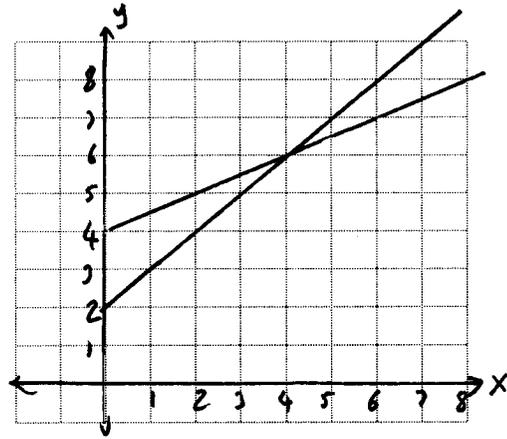
APPENDIX D

Interview-4 Questions

1. The following graph shows the temperature in two cities. Compare the change of temperature in the two cities.



2. The following graph shows the growth of two babies. Compare their growth.



3. Linda's savings can be described by the equation $y=300-10x$, where x stands for the weeks and y stands for the money.

Adam's savings can be described by the following table:

Weeks	0	1	2	3	4
Money	200	195	190	185	180

Compare the savings of Linda and Adam. Who has a larger amount at the beginning of the year? Is their money going to be equal in the future? If yes, when?

APPENDIX E
Class Worksheets

APPENDIX E1

Name _____

Date _____

Find the rule that relates x and y

1.

x	0	1	2	3	4
y	6	7	8	9	10

$y =$ _____

2.

x	0	1	2	3	4
y	0	-2	-4	-6	-8

$y =$ _____

3.

x	0	1	2	3	4
y	1	3	5	7	9

$y =$ _____

4.

x	0	1	2	3	4
y	1	2	5	8	11

$y =$ _____

5.

x	0	1	2	3	4
y	0	9	18	27	36

$y =$ _____

6.

x	0	1	2	3	4
y	1	-4	-9	-14	-19

$y =$ _____

Match each table with two equivalent equations.

1.

x	y
-1	-2
0	0
2	4
5	10

a. $x - y =$ _____

b. $2y = 4x$

2.

x	y
-4	-4
-1	-1
3	3
6	6

c. $y = x + 1$

d. $y = 2x$

e. $y = x$

f. $x - y = 7$

3.

x	y
-3	-2
-1	0
0	1
2	3

Name _____

DATE: _____

PERIOD: _____

Linear Functions

Given the following function rules, complete the table of values.

1. $y = 6x - 8$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

2. $y = -2x + 5$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

3. $y = -5x + 2$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

1. $y = 4x - 6$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

2. $y = -3x + 3$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

3. $y = -4x + 2$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

1. $y = 6x + 2$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

2. $y = -7x + 5$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

3. $y = -5x - 3$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

1. $y = 8x - 7$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

2. $y = -6x + 8$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

3. $y = -3x - 5$

Input (x)	Output (y)
-1	
0	
1	
2	
3	

APPENDIX E2

Name _____

**Practice
4-1**

Understanding Two-Variable Relationships

Find the value of y when $x = 7$ in each of the following equations.

1. $y = 10x$

2. $y = x - 4$

3. $y = 14x$

4. $y = x + 23$

Complete each table of values.

5.

x	0	1	2	3	4	5
$y = x + 5$						

6.

x	0	1	2	3	4	5
$y = 4x$						

7.

x	0	1	2	3	4	5
$y = x - 7$						

8.

x	0	1	2	3	4	5
$y = -8x$						

Make a table of values for each equation. Use 0, 1, 2, 3, 4, and 5.

9. $y = x + 10$

10. $y = -3x$

Find a rule that relates x and y in each table. Then find y when $x = 25$.

11. rule: _____ 12. rule: _____ 13. rule: _____

$y =$ _____ $y =$ _____ $y =$ _____

x	y
0	0
1	-7
2	-14
3	-21
4	-28
5	-35

x	y
0	7
1	8
2	9
3	10
4	11
5	12

x	y
0	0
1	11
2	22
3	33
4	44
5	55

14. **Science** Every spider has 8 legs. Make a table relating the number of spiders to the number of legs.

Name _____ Date _____

Linear Equations With Two Variables

You have learned that variables are used in equations to represent numbers. Variables are represented in equations with letters. The equation $x = y + 3$ can be used to demonstrate how variables are used in equations.

$x = y + 3$	If $x = 7$, then y must equal 4. If $x = 10$, then y must equal 7. If $x = 40$, then y must equal 37. If $x = -4$, then y must equal -7.
-------------	---

In $x = y + 3$, the value of y depends on the value assigned to x . The variable is y , since it may have different values.

The equation $x = y + 3$ is a linear equation. A linear equation will be a straight line when graphed on a rectangular coordinate system.

Many linear equations will be in a form like $2x + y = 16$.

$2x + y = 16$	If $x = 4$, then y equals 8. If $x = 3$, then y equals 10. If $x = 7$, then y equals 3.
---------------	--

Solve the following (use the values for the variables indicated to solve each equation).

1. $x + y = 12$ What does y equal if $x = 3$? _____ 4? _____ 9? _____
2. $2x + y = 20$ What does y equal if $x = 7$? _____ 5? _____ 8? _____
3. $7 + x = y$ What does y equal if $x = 12$? _____ 10? _____ 1? _____
4. $5x + y = 30$ What does y equal if $x = 10$? _____ 8? _____ 7? _____
5. $3x + 2y = 12$ What does y equal if $x = 2$? _____ 1? _____ 0? _____
6. $7 + 2y = x$ What does y equal if $x = 9$? _____ 11? _____ 23? _____
7. $6x - 2 = y$ What does y equal if $x = 1$? _____ 3? _____ -2? _____
8. $4x - 3 = y$ What does y equal if $x = -1$? _____ -2? _____ -3? _____

Name _____

**Practice
4-2**

Solutions of Two-Variable Equations

Determine whether each ordered pair is a solution of the equation.

- | | | | |
|------------------------|------------|-------------|--------------|
| 1. $y = x + 9$ | a. (4, 13) | b. (5, -4) | c. (-8, 1) |
| _____ | _____ | _____ | _____ |
| 2. $y = 4x$ | a. (3, 12) | b. (16, 4) | c. (6, 10) |
| _____ | _____ | _____ | _____ |
| 3. $y = -\frac{1}{3}x$ | a. (-3, 9) | b. (6, -2) | c. (-15, -5) |
| _____ | _____ | _____ | _____ |
| 4. $x = y + 4$ | a. (-2, 2) | b. (-2, -6) | c. (12, 8) |
| _____ | _____ | _____ | _____ |
| 5. $x + 2y = 10$ | a. (2, 8) | b. (4, 3) | c. (3, 4) |
| _____ | _____ | _____ | _____ |

Give two solution pairs for each equation.

- | | | |
|-----------------|----------------------------|-------------------|
| 6. $y = x + 8$ | 7. $y = x - 3$ | 8. $y = 7x$ |
| _____ | _____ | _____ |
| 9. $y = 3x - 2$ | 10. $y = \frac{1}{4}x + 8$ | 11. $y = -3x + 7$ |
| _____ | _____ | _____ |
| 12. $x - y = 5$ | 13. $2x + y = 11$ | 14. $x - 4y = 10$ |
| _____ | _____ | _____ |

15. The equation $y = 0.23x + 0.09$ gives the cost, in dollars, of mailing a letter weighing x ounces, where x is a positive integer. Make a table showing the number of ounces and the price, for letters weighing 1 to 6 ounces.

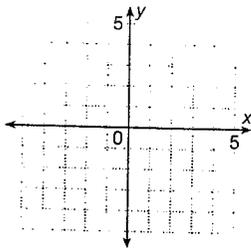
Name _____

**Practice
4-3**

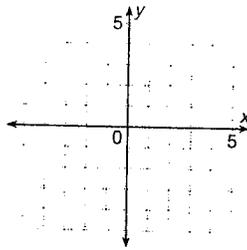
Graphing Two-Variable Relationships

Graph each equation. Use 0, 1, 2, and 3 as x-values.

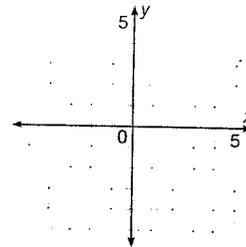
1. $y = x - 2$



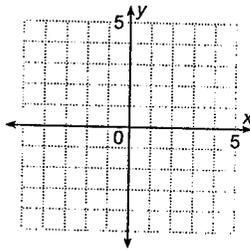
2. $y = -\frac{1}{3}x$



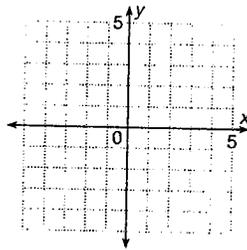
3. $y = -x + 1$



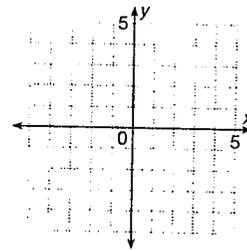
4. $y = \frac{1}{2}x + \frac{3}{2}$



5. $y = 3x - 4$



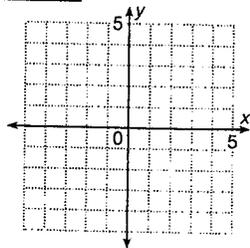
6. $y = -2x + 3$



Graph the ordered pairs in each table. Connect the points to determine if the graphs are linear. Write linear or not linear.

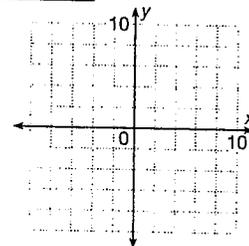
7. _____

x	y
-3	3
-1	2
1	1
4	0

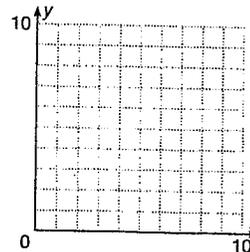


8. _____

x	y
-6	2
-4	3
0	5
6	8



9. **Consumer** Adeshima plans to order some fabric from a mail-order catalog. The price is \$0.75 per yard, plus \$2.00 for shipping. Use x for the number of yards, and y for the price she will pay. Graph the price that she will pay.



APPENDIX E3

Name _____

**Alternative
Lesson
10-2**

Linear Functions

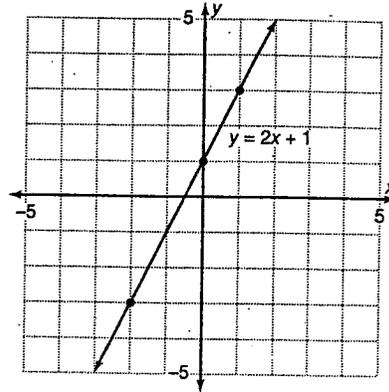
If we let x represent the input value and y represent the output value in a function, then the output, y , depends on the input, x . In a function, the variable y is called the **dependent variable**, and the variable x is called the **independent variable**. A function that can be graphed as a straight line is called a **linear function**.

Example 1

Graph the equation $y = 2x + 1$. Does it describe a function? Is it a linear function?

First, make a table of input and output values. Then graph the ordered pairs (x, y) .

Input (x)	Function Rule $2x + 1$	Output (y)
-2	$2(-2) + 1$	-3
0	$2(0) + 1$	1
1	$2(1) + 1$	3
2	$2(2) + 1$	5

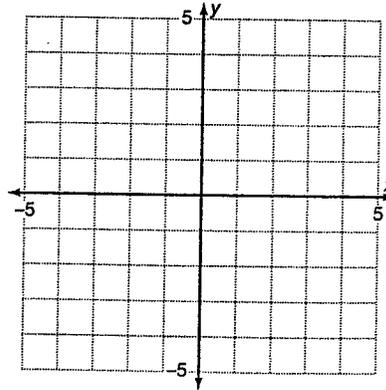


The equation $y = 2x + 1$ is a function. Each input (x) value has a single output (y) value. It is a linear function because the graph is a line.

Try It Graph the equation $y = -2x + 3$.

a. First, make a table of input and output values.

Input (x)	Function Rule $-2x + 3$	Output (y)
-1	$-2(-1) + 3$	
0	$-2(0) + 3$	
1		



b. Graph the ordered pairs.

c. Does each input (x) value have a single output (y) value? _____

d. Is it a function? _____

e. Is the graph a line? _____

f. Is it a linear function? _____

g. Label your graph.

On the same grid, graph the equation $y = x + 1$. Label your graph.

h. Is it a linear function? How do you know? _____

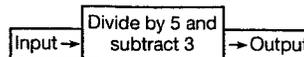
Name _____

**Practice
10-1**

Functions

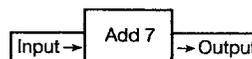
For the function machine shown, find the output value for each input value.

1. Input of 20 _____ 2. Input of 5 _____
 3. Input of -35 _____ 4. Input of 13 _____



For the function machine shown, find the input value for each output value.

5. Output of 6 _____ 6. Output of 12 _____
 7. Output of -28 _____ 8. Output of -13 _____



What is a possible rule for the input and output shown in each table?

9.

Input	1	3	5	9
Output	16	18	20	24

10.

Input	-1	0	1	2
Output	-5	-2	1	4

11.

Input	-8	-3	2	7
Output	7	7	7	7

12.

Input	64	4	-4	-8
Output	16	1	-1	-2

13. At Roy's Donut Shop, if you buy 1 to 4 donuts, you pay \$0.50 per donut. If you buy 5 to 8 donuts, the price is \$0.40 each. Roy limits each customer to 8 donuts.

- a. Complete the table by finding the *total* purchase price for each number of donuts.

Number of donuts	0	1	2	3	4	5	6	7	8
Total price (\$)									

- b. Is the price a function of the number of donuts? Explain. _____

 c. Is the number of donuts a function of the price? Explain. _____

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BIOGRAPHICAL SKETCH

Evrin Erbilgin was born in Fethiye, Mugla, Turkey in 1978. She graduated from Middle East Technical University in Ankara, Turkey in 1999. Then, she had two years of teaching experience: one year in TED Ankara College as a mathematics teacher, and one year in Selcuk Ozsoy High School as a mathematics teacher. In 2001 she came to the USA to work on her graduate education. Currently, she is completing her masters in mathematics education at Florida State University.