2011

Sequences of Pseudo-Anosov Mapping Classes with Asymptotically Small Dilatation

Aaron David Valdivia Jr.

THE FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

SEQUENCES OF PSEUDO-ANOSOV MAPPING CLASSES WITH ASYMPTOTICALLY SMALL DILATATION

By

AARON DAVID VALDIVIA JR.

A Dissertation submitted to the
Department of Mathematics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Degree Awarded:
Fall Semester, 2011

Aaron David Valdivia Jr. defended this dissertation on September 14, 2011.

The members of the supervisory committee were:

Eriko Hironaka
Professor Directing Dissertation

Laura Reina
University Representitive

Wolfgang Heil
Committee Member

Eric Klassen
Committee Member

The Graduate School has verified and approved the above-named committee members, and certifies that the dissertation has been approved in accordance with the university requirements.

To my parents, Aaron and Kathleen, for support through everything.

ACKNOWLEDGMENTS

I would like to thank my advisor, Eriko Hironaka, for the topic of my thesis and for working with me the past three years trying to make my work the best it could be. Also thanks to Chai-Yen Tsai for discussing the earlier version of the results in this thesis with me. Finally thanks to my friends who I won't name for fear of leaving someone out, you all made it so much fun.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

Given a surface, $S_{g,n}$, of finite genus and punctures, a mapping class, ϕ, is said to be pseudo-Anosov if it preserves a pair of transverse projective measured laminations on the surface, one expanding and one contracting, called unstable and stable respectively. The dilatation $\lambda(\phi) > 1$ is the expanding factor of the unstable lamination. Penner showed that the minimal dilatation for pseudo-Anosov elements on closed surfaces is asymptotic to $\frac{1}{|\chi(S_{g,n})|}$ where $\chi(S_{g,n})$ is the Euler characteristic of the surface. We generalize Penner's proof and show if the genus and punctures are related by any of the linear equations $g = r(n-b)$ where $r > 0$ is rational and $b = 0, 1, 2$, then the minimal dilatation is asymptotic with $\frac{1}{|\chi(S_{g,n})|}$. We show that certain sequences of mapping classes have homeomorphic mapping tori, and give evidence that for fixed number of punctures the minimal dilatation behaves asymptotically like $\frac{1}{|\chi(S_{g,n})|}$.

CHAPTER 1

INTRODUCTION

In this thesis we study the minimal dilatation problem for surface homeomorphisms. We present a construction which gives sequences of homeomorphisms exhibiting small dilatation with respect to Euler characteristic, and that have homeomorphic mapping tori. This generalizes a construction of Penner's which he used to give a bound for the minimal dilatation, $\delta_{g,n}$, on closed surfaces.

The mapping class group is the group of isotopy classes of orientation preserving homeomorphisms. By the Nielson-Thurston classification, the elements of the group are either periodic, reducible, or pseudo-Anosov. A mapping class, ϕ, is called pseudo-Anosov if the length of a geodesic grows exponentially by a factor $\lambda(\phi) > 1$ called the dilatation of ϕ. For each surface there is a minimal dilatation bounded away from 1 [Yoc81] [Iva90]. One question related to the minimal dilatation is the following:

Question 1.0.1. *What is the asymptotic behavior of a sequence of numbers* $\log(\delta_{g_i,n_i})$ *with* $i \in \mathbb{N}$*? [Tsa09]*

From our construction we find the asymptotic behavior when the genus and punctures are related by certain linear equations.

Theorem 1.0.2. *If we are given a rational number* $r > 0$ *and* $b = 0, 1,$ *or* 2 *then*

$$\log(\delta_{g_i,n_i}) \asymp \frac{1}{|\chi(S_{g_i,n_i})|}.$$

where (g_i, n_i) *satisfy the equation* $g_i = r(n_i - b)$*.*

Further, our construction yeilds sequences of mapping classes ϕ_m which we call Penner sequences (See Chapter 3).

Theorem 1.0.3. *Given a Penner sequence,* ϕ_m*, there is a single homeomorphism class for the mapping tori.*

The thesis is organized as follows: Chapter 2 gives background on our methods and techniques. We discuss foliations in order to talk about the singularity data of a pseudo-Anosov mapping class and about when these mapping classes extend to surfaces with fewer punctures. We define laminations as a more intuitive way of thinking of foliations and to help in defining and understanding train tracks, our main tool. Train tracks will be used to find dilatations of mapping classes and for examining singular data. Lastly, we discuss 3-manifolds, mapping tori, and give algorithms for computing certain invariants related to dilatations. In Chapter 3 we present our construction and explore the properties that these examples have. We also show how to apply our construction in proving our main theorem. In Chapter 4 we present some specific examples of our construction to better illustrate the proofs as well as to examine their mapping tori in terms of their Thurston norms and other fibrations they admit. In the appendix we present a proposed method for finding asymptotic behavior for other rays and partial results obtained from the magic manifold.

CHAPTER 2

BACKGROUND

In this Chapter we provide the motivation for our results and background for the methods we will use. We discuss foliations in order to give precise definitions and to talk about when mapping classes on punctured surfaces extend to mapping classes on surfaces with fewer boundary components and laminations in order to give a better understanding of train tracks and how we use them to find dilatations. We discuss train tracks and Perron-Frobenius matrices as these are our main tools for finding and estimating dilatations and to identify when we can fill punctures on surfaces without effecting the dilatation of the induced mapping class. We also include some background on 3-manifolds especially how they relate to the mapping tori of pseudo-Anosov mapping classes. We include methods for computing Alexander and Teichmuller polynomials of these manifolds which allow us to compute the fibered faces that the mapping classes lie in.

2.1 Motivation

One problem related to minimal dilatation pseudo-Anosov mapping classes is to compute the minimal dilatation for a given surface $S_{g,n}$. This has been done for only a handful of cases. The case for the once punctured torus is well known and is the largest root of the matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

The minimal dilatation elements for spheres with less than 6 punctures was found by [Ham06] [HS07]. The minimal dilatation for the closed genus 2 surface was found by [CH08]. For pseudo-Anosovs whose invariant foliations are orientable the minimal dilatation is known for closed surfaces of genus 2, 3, 4, and 5 [EL11], for genus 7 by [KT10] and [AD10], and genus 8 by [Hir10]. More is known about bounds for these numbers and their asymptotic behavior. In this thesis we do not attempt to find minimal dilitation elements but to find bounds for them by finding new examples of pseudo-Anosov mapping classes. Penner proved the lower bound for arbitrary surfaces of negative Euler characteristic to be

$$\log(\delta_{g,n}) \geq \frac{\log(2)}{12g-12+4n} \text{ [Pen91]}.$$

Further he defined a class of examples for closed surfaces $S_{g,0}$ and used them to give the behavior for the minimal dilatations, $\delta_{g,0}$, which he showed is

$$\log(\delta_{g,0}) \asymp \frac{1}{|\chi_{g,0}|} \text{ [Pen91]}.$$

The examples Penner gives for the upper bound are contained in the new examples we present. Penner's results on asymptotics have been extended by Hironaka and Kin [HK06] to punctured spheres:

$$\log(\delta_{0,n}) \asymp \frac{1}{|\chi(S_{0,n})|}.$$

Continuing the investigation of asymptotics for minimal dilatations Tsai [Tsa09] considered the gn-plane to be the integer lattice in the first quadrant where each point (n, g) is assigned the the number $\log(\delta_{g,n})$. Then for a rational ray one can consider the asymptotic behavior of the numbers associated to the integer points on the ray. Tsai found that the horizontal rays in this plane exhibit a different behavior:

$$\log(\delta_{g,n}) \asymp \frac{\log(|\chi(S_{g,n}|))}{|\chi(S_{g,n})|} \text{ where } g \text{ is fixed } g > 1.$$

In her paper Tsai asked what is the asymptotic behavior of other rays in the gn-plane? Our Theorem gives the asymptotics for all rational rays of positive slope passing through the origin, $(1, 0)$ and $(2, 0)$.

Further inspiration for this thesis comes from work of Farb, Leininger and Margalit in [LM09].

Question 2.1.1. *What is the shape of the mapping tori for "small" dilatation examples?*

Farb, Leininger, and Margalit define:

$$\Lambda_P = \{\phi : S_{g,n} \to S_{g,n} \mid \bar{\lambda}(\phi) \leq P\}.$$

Then they restrict the mapping classes to the nonsingular points of the transverse foliations defined by the pseudo-Anosov calling the surface with the singular points removed $\tilde{S}_{g,n}$.

$$\tilde{\Lambda}_P = \{\tilde{\phi} : \tilde{S}_{g,n} \to \tilde{S}_{g,n} \mid \phi \in \Lambda_P \text{ and } \tilde{\phi} \text{ is the restriction of } \phi \text{ to the surface } \tilde{S}_{g,n}\}.$$

Denote by M_ϕ the mapping torus for the mapping class ϕ.

Theorem 2.1.2. *[LM09]*

Let

$$T_P = \{M_{\tilde{\phi}} \mid M_{\tilde{\phi}} \textit{ is homeomorphism class of a mapping torus for } \tilde{\phi} \in \tilde{\Lambda}_P\}.$$

For any $P > 1$ the set T_P is finite.

In this thesis we show that certain small dilatation examples come from a single 3-manifold. Each of these manifolds admits various fibrations each with pseudo-Anosov monodromy. The theory of fibered faces and Teichmuller polynomials defined in [McM99] produces a way to study the other fibrations of these manifolds.

The question of which three manifolds are obtained from minimal dilatation elements changes the question of what are the minimal dilatations, to what are the minimal 3-manifolds? There is not much that is known about what these manifolds look like. Agol gives a bound on the number of 3-manifolds that are obtained as mapping tori for minimal elements on closed surfaces in [Ago10]. In this thesis we consider the magic manifold which has been shown to contain many low dilatation examples. It also is the candidate for the least volume hyperbolic 3-manifold with 3-cusps [KT09]. One reason the magic manifold has been given such consideration is because of the relations between low volume and low dilatation [EKT09]. Their results relating volume of mapping tori and dilatation give an idea of what minimal dilatation mapping class's mapping tori must look like.

2.2 Foliations and laminations

Foliations will be used later to identify which mapping classes extend nicely to surfaces with fewer punctures and to define pseudo-Anosov mapping classes in another way. The theory of laminations give an understanding of how train tracks, defined in a later section, work.

Definition 2.2.1. A *singular foliation* on a surface S is a partition of a surface $S\backslash P$ into real lines, $(-\infty, \infty)$, called *leaves* where P is a finite set of points. The finite points are called the *singular points*. Non-singular points are called *regular* and there are local charts that are modeled on the pictures in Figure 2.1.

Singular points are said to be n-pronged if there are n-leaves emanating from the point; regular points are said to be 2-pronged. Punctures are also said to be p-pronged if they have local pictures looking like punctured singular points.

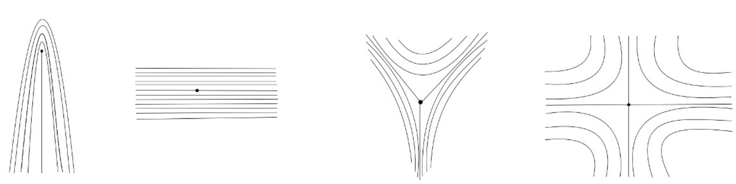

Figure 2.1: Neighborhoods of points on singular foliations

Figure 2.2 gives an example of a singular foliation on the pair of pants, the sphere with 3 boundary components, with two 3-pronged singularities.

For some singular foliations we can define a measure on the transverse arcs to the foliation called a transverse measure.

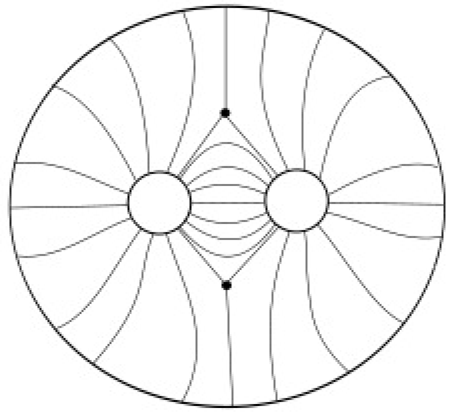

Figure 2.2: Measured foliation

Definition 2.2.2. A *transverse measure* μ is a measure on the arcs transverse to a singular foliation $\mathcal{F}$ such that for all arcs α and α' transverse to $\mathcal{F}$

- 1. If $\alpha \subset \alpha'$ then $\mu(\alpha) \leq \mu(\alpha')$.
- 2. If α is homotopic to α' relative to the leaves of $\mathcal{F}$ then $\mu(\alpha) = \mu(\alpha')$.

Definition 2.2.3. A *measured foliation* is a pair $(\mathcal{F}, \mu)$ where $\mathcal{F}$ is a singular foliation and μ is a transverse measure for $\mathcal{F}$.

Example: Transverse measures on the singular foliation given in figure 2.2, can be defined as follows: The given example consists of 3 regions that can be thought of as rectangles where the leaves are the horizontal cross sections of each rectangle. If we choose a height for each rectangle then the transverse measure assigns to each transverse arc the sum of the Euclidean lengths in the vertical direction for each sub arc passing through the rectangles. So any assignment of positive value for each of these regions gives a measured singular foliation. This is a model for one of 6 possible measured foliations on the pair of pants and all measured foliations on closed surfaces may be constructed from these 6 models for the pair of pants. More details can be found in [bDKM11]. The idea of a foliation leads naturally to the definition of laminations.

Definition 2.2.4. A *lamination* of a surface is a C^1 foliation of a closed subset of the surface. By C^1 we mean that this foliation has no singular points.

Definition 2.2.5. A *measured lamination* (L, λ) is a lamination L together with a measure on the transverse 1-submanifolds whose boundary, if nonempty, lies outside of L. We call the measure λ and the set of 1-submanifolds Λ. The measure must be countably additive, invariant under homotopy where each level of the homotopy is an element of Λ, and $\lambda(\alpha) = 0$ if and only if $\alpha \cap L = \emptyset$.

Example: Weighted curve systems form a class of examples of measured laminations. Consider a multicurve $C = \sqcup_{j=1}^{n} S_j^1$ embedded in a surface S. Associate to each curve S_j a weight w_j.

Then the lamination is just the multicurve C and the measure of a transverse arc t is given by $\lambda(t) = \sum_{j=1}^{n}(w_j i(S_j, t))$, where $i(S_j, t)$ is the minimal intersection number of S_j and t over homotopy classes of t contained in Λ.

Theorem 2.2.6. *[PH92] There is an identification between the space of measured singular foliations and the space of measured laminations of a surface.*

The proof involves a process called enlargement which can be found in [bDKM11], which allows us to consider laminations instead of singular foliations. We want to consider these laminations for the combinatoral train track approach which we will discuss later.

2.3 The mapping class group and pseudo-Anosov mapping classes

Denote by $Homeo^+(S_{g,n})$ the group of orientation preserving homeomorphisms on the surface $S_{g,n}$ with genus g and n punctures. Then $Homeo_0(S_{g,n})$ is the subgroup of homeomorphisms isotopic to the identity map.

Definition 2.3.1. The *mapping class group* of the surface $S_{g,n}$ is the quotient

$$Mod(S_{g,n}) = Homeo^+(S_{g,n})/Homeo_0(S_{g,n}).$$

Definition 2.3.2. An element $\phi \in Mod(S_{g,n})$ is called *pseudo-Anosov* if it contains a representative that fixes a pair of transverse projective singular measured foliations i.e.

$$\phi(\mathcal{F}^{\pm}, \mu^{\pm}) = (\mathcal{F}^{\pm}, \lambda^{\pm 1}\mu^{\pm}).$$

The real number $\lambda > 1$ is called the *dilatation*.

Elements of the mapping class group can be classified in the following way.

Theorem 2.3.3. *[bDKM11, Nielsen Thurston Classification] An element $\phi \in Mod(S_{g,n})$ is one of the following:*
1. periodic;
2. reducible ($\phi(c) = c$, where c is a 1-submanifold of $S_{g,n}$); or
3. pseudo-Anosov.

The proof can be found in [bDKM11, expose 9]. Essentially one can show that the Teichmuller space is homeomorphic to a ball and Thurston shows that the boundary can be thought of as the space of projective measured laminations of the surface. Then a mapping class has a continuous action on this closed space and by Brouwer fixed point theorem has at least one fixed point. The theorem is proved by analyzing the different types of fixed points this action may have.

An idea that we will use later is when a pseudo-Anosov mapping class on a surface with punctures extends to another pseudo-Anosov mapping class with the same dilatation on a surface with fewer punctures.

Lemma 2.3.4. *If ϕ is a pseudo-Anosov mapping class on the surface $S_{g,n}$, some subset of the punctures I is fixed setwise, and if none of the points in I are 1-pronged then the punctures may be filled in and the induced mapping class $\tilde{\phi}$ is pseudo-Anosov with $\lambda(\phi) = \lambda(\tilde{\phi})$.*

Proof: If none of the punctures in I are 1-pronged then after filling in the punctures of the original transverse foliations we have transverse foliations which are fixed under the induced mapping class on the filled in surface.□

The mapping class group can be generated by maps defined by curves in the surface called Dehn twists.

Definition 2.3.5. Given a simple closed curve c on a surface $S_{g,n}$ the *Dehn twist* $\tau_c : S_{g,n} \to S_{g,n}$ is defined by the identity outside of an annulus $I \times S^1$ about the curve c and by the map

$$(t, e^{i\theta}) \to (t, e^{2\pi i t + \theta})$$

for $0 \leq t \leq 1$ and $0 \leq \theta < 2\pi$.

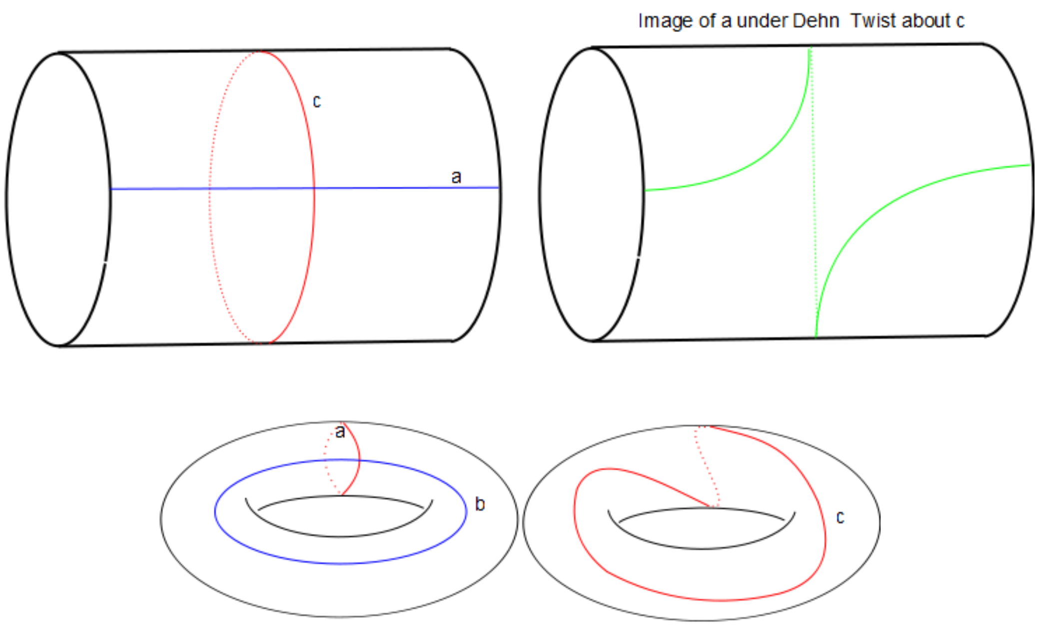

Figure 2.3: Dehn Twists

Figure 2.3 shows a Dehn twist in an annulus and an example of curves a and b and the image of a under a Dehn twist about b. Dehn showed that the mapping class group of a closed surface is generated by finitely many Dehn twists. Since we are interested in pseudo-Anosov mapping classes we want to know how we can find pseudo-Anosov elements as compositions of Dehn twists. Given a set of generators in the mapping class group we would like to know which words of Dehn twists give pseudo-Anosov mapping classes.

Theorem 2.3.6. *[Pen88, Lemma 3.1, Penner's Semigroup Criteria] Suppose $\mathcal{C}$ and $\mathcal{D}$ are each disjointly embedded collections of simple closed curves in an oriented surface S. Suppose $\mathcal{C}$ interesects $\mathcal{D}$ minimally and $\mathcal{C} \cup \mathcal{D}$ fills S(i.e. the connected components of $S/C \cup D$ have non-negative Euler characteristic). Let $R(\mathcal{C}^+, \mathcal{D}^-)$ be the semigroup generated by $\{\tau_c \mid c \in \mathcal{C}\} \cup \{\tau_d^{-1} \mid d \in \mathcal{D}\}$. If $\omega \in R(\mathcal{C}^+, \mathcal{D}^-)$, then ω is pseudo-Anosov if each τ_c and τ_d^{-1} appear in ω.*

From now on we will refer to this theorem as Penner's semigroup criteria.

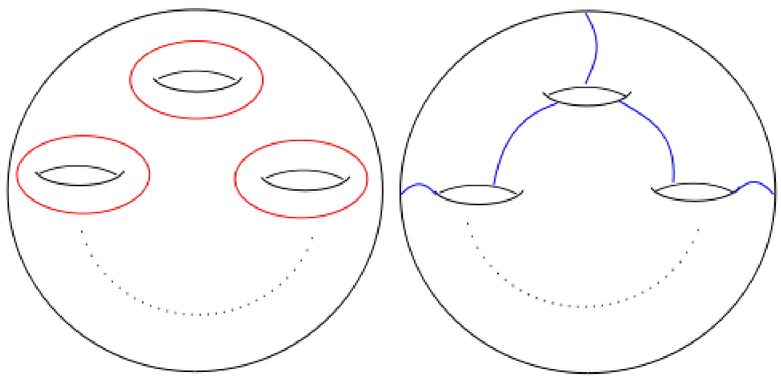

Figure 2.4: Multitwists for semigroup criteria

Example: The two multicurves in Figure 2.4 satisfy Penner's semigroup criteria. Thus any word in the semigroup generators that contains at least one copy of each generator defines a pseudo-Anosov mapping class.

2.4 Train tracks and dilatations

One tool for studying pseudo-Anosov maps is train tracks. Train tracks give a combinatoral way to view measured foliations of surfaces and find the dilatation of pseudo-Anosov mapping classes. We define Perron Frobenius matrices and explain how we will use them to bound dilatation. Last we show how to find the singularity data of a pseudo-Anosov mapping class from a train track.

Definition 2.4.1. A graph τ embedded in a surface S with edges (called *branches*) and vertices (called *switches*) is called a *train track* if the following hold.
1. The embedding is C^1 away from the switches and there is a tangent direction at each switch so that a C^1 path parameterized by $\phi : (0,1) \to \tau$ with $\lim_{t \to 1} \phi(t) = v$ extends to $(0,1]$ where $\phi(1)$ is a switch v and the one sided derivative lies in the tangent line to the point.
2. There exists a smooth path through each switch v

$$\phi : (0,1) \to \tau$$

such that $\phi(\frac{1}{2}) = v$.

3. The double of each component of $S \backslash \tau$ along the smooth part of the boundary must have negative Euler characteristic.

Figure 2.5 is an example of a train track with 3 switches and 5 edges on a 4 punctured sphere.

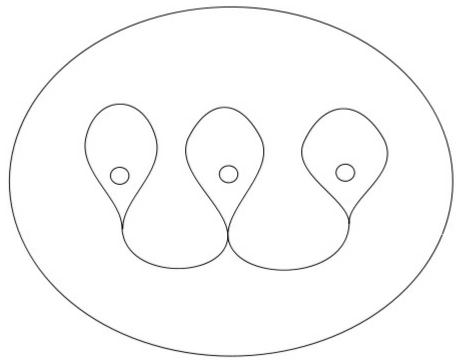

Figure 2.5: A train track

Figure 2.6 is an example of a similar graph which fails to be a train track for more than one reason. There is not a definable tangent direction at the middle switch and the middle monogon has a disc as a double along the smooth boundary which has positive Euler characteristic

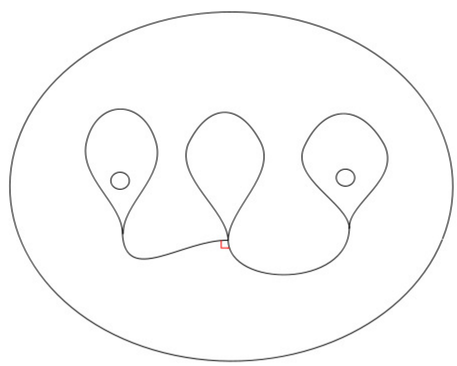

Figure 2.6: Fails to be a train track

Definition 2.4.2. A smoothly embedded multicurve or a train track C in S is *carried* by a train track τ if there is a map $\phi : S \to S$ called the *carrying map* such that the following hold.

1. The image $\phi(C)$ is contained in τ.
2. The map ϕ is isotopic to the identity.
3. The map $\phi \mid_C$ is an immersion.

For example given the train track above on the 4 punctured sphere, a curve parallel to any boundary component is not carried by the train track. However, a curve about the left and middle boundary components is carried by the train track.

Theorem 2.4.3. *[PH92] Given a pseudo-Anosov mapping class ϕ on a surface S we can find a train track τ such that $\phi(\tau)$ is carried by τ.*

The concept of carrying leads to a measure on the train track where given a carrying of $\phi(C)$ each edge of τ is assigned a weight equal to the cardinality of the preimage of a point on the edge under $\phi\,|_C$.

Definition 2.4.4. A *transverse measure* on a train track τ is a function μ which assigns a real number to each edge of τ such that the sum of the weights on the ingoing edges is equal to the sum of the weights on the outgoing edges at each switch. Ingoing and outgoing are defined by the chosen tangent direction at the switch. The requirements at each switch are called the *switch conditions*.

The above definition says that a switch looking like the one in Figure 2.7 would have to have weights a, b, and c such that $a = b + c$.

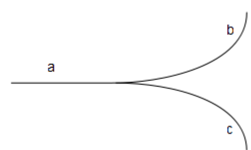

Figure 2.7: Admissable measure

We want to consider the positive transverse measures of a train track. These correspond to a cone in $\mathbb{R}^E$ where E is the number of edges on the train track. The measures also correspond to measured laminations on the surface. As we have already mentioned, laminations correspond to measured singular foliations. If we find a train track σ whose image under the mapping class in question is carried by itself, i.e. $\phi(\sigma)$ is carried by σ, then we can consider the action of the mapping class on the space of transverse measures for this train track.

Assume we have a mapping class $\phi : S \to S$ and a train track $\sigma \subset S$ such that $\phi(\sigma)$ is carried by σ. Say σ has E edges. Let $\mathbb{R}^E$ be the vector space corresponding to all of the possible weights on edges of σ. A standard basis vector b_i corresponds to a weight of 1 on the edge e_i. The action of the mapping class ϕ depends on the carrying map. Once we have chosen a carrying map the action takes the vector b_i to $\sum_1^E n_j b_j$ where n_j is the number of disjoint arcs of b_i carried by the edge b_j. This defines a linear map on the set of all measures and it is easy to see that the subspace of admissable measures is invariant with respect to this action.

Finding the real eigenvectors of this transition matrix is equivalent to finding invariant laminations. In particular a strictly positive eigenvector gives a positive weight for each edge. If we attach rectangles to each edge of width equal to the weight given and contract the boundaries we have constructed the expanding foliation. Further the eigenvalue determines the amount that the width of these rectangles are stretched under each iteration. Thus it equals the dilatation of the mapping class.

Example: Given an integral transverse measure for a train track one can construct a multicurve associated to the measure assigning the edge i the weight k_i. Draw k_i arcs parallel to the edge i and connect the arcs in a way that gives a closed multicurve. Then assigning each individual curve with the weight of 1 we have a weighted curve system associated to the transverse measure.

Definition 2.4.5. A *Perron-Frobenious matrix* is one with entries $a_{i,j} \geq 0$ such that some power of the matrix has strictly positive entries.

Theorem 2.4.6. *A Perron-Frobenious matrix has a unique positive eigenvector, $\mid x \mid = 1$, with real eigenvalue $\lambda > 1$ equal to the spectral radius.*

Theorem 2.4.7. *The spectral radius of a Perron-Frobenius matrix is bounded by the largest column sum.*

Proof: Let M be a Perron-Frobenius matrix with spectral radius λ and corresponding eigenvector v with norm 1. This eigenvector is real and positive.

$$\mid vM \mid = \lambda = \frac{\sum_{i=1}^{n} v_i m_{ij}}{v_j} \text{ for all } j = 1...n.$$

So then we pick the largest component of v say v_j. Then we have:

$$\mid vM \mid = \lambda = \sum_{i=1}^{n} \frac{v_i}{v_j} m_{ij}.$$

But each term $\frac{v_i}{v_j} \leq 1$ and each term $m_{ij} \geq 0$ and so we have the inequality:

$$\lambda \leq \sum_{i=1}^{n} m_{ij}.$$

We don't know which j we picked and so we bound it by the largest column sum.□

Theorem 2.4.8. *[PR87, Lemma 4.1] Given a pseudo-Anosov mapping class there exists a train track τ such that the matrix M which determines the action on the transverse measures is Perron-Frobenious for some choice of collapsing map. The positive eigenvector determines an invariant measure corresponding to the invariant foliation and the eigenvalue is the dilatation.*

These transition matrices will be the tool we use to find bounds for the dilatations of pseudo-Anosov mapping classes that we construct using Penner's semigroup criteria.

Lemma 2.4.9. *Given a pseudo-Anosov mapping class on S and an invariant train track τ m-gons in $S \backslash \tau$ are in $1-1$ correspondence with the set of m-pronged singularities and punctures.*

Proof: The pseudo-Anosov determines an invariant projective measure on the train track. From this measure we may construct the a foliation on the surface. Given a weight for an edge of the train track we glue a rectangle to the edge and foliate the rectangle with leaves parallel to the edge we glued to. Define the width of this rectangle to be the weight of the edge. Performing this operation on all edges and then identifying the boundary components gives the measured foliation on the surface with a m-pronged singularity for each m-gon.□

Using this theorem we can find sets of singularities which are not one pronged and obtain induced mapping classes on filled in surfaces with the same dilatation.

2.5 3-Manifolds

We are interested in 3-manifolds which fiber over the circle. Mapping tori for pseudo-Anosovs which have more than one cusp fiber in a variety of ways. Each fibration determines new examples of pseudo-Anosov mapping classes with different dilatation. We discuss The theory of fibered faces, a function on the fibered face giving the dilatation of the pseudo-Anosovs, and methods for computing Alexander and Teichmuller polynomials which allow us to compute a fibered face of a 3-manifold.

Definition 2.5.1. A *fibration* consists of the data (F, E, π, B) where F is the *fiber*, E is the *total space*, B is the *base space*, and π is a continuous surjection from the total space to the base space. Further there is an open neighborhood U about a point in the base space such that $\pi^{-1}(U)$ is homeomorphic to $F \times U$.

We will only consider fiber bundles with base space the circle. These bundles are called *manifolds fibered over the circle*. One way to view examples of these is to construct mapping tori for a homeomorphism ϕ. The mapping torus for $\phi : S \to S$ is defined by:

$$M_\phi = I \times S/(1, x) \sim (0, \phi(x)).$$

For a given pseudo-Anosov mapping class its mapping torus gives us an associated 3-manifold. For $P \in \mathbb{R}$, $P > 1$ define:

$$\Psi(P) := \{\phi \mid \chi(S) < 0,\, \phi : S \to S,\, \lambda(\phi) \leq P^{\frac{1}{|\chi(S)|}}\}$$

where $\chi(S)$ denotes the Euler characteristic of S and $\lambda(\phi)$ denotes the dilatation of ϕ. Then denote by $S^\circ_\phi = S^\circ$ the surafce obtained after removing the singularities from the invariant foliation.

$$\Psi^\circ(P) = \{\phi : S^\circ \to S^\circ \mid \phi \in \Psi(P)\}$$

Then we denote $\mathcal{T}(\Psi^\circ(P))$ as the homeomorphism classes of mapping tori for $\phi \in \Psi^\circ(P)$. Theorem 2.1.2 concerns the finiteness of this set.

We will construct classes of examples and show they have homeomorphic mapping tori once we remove the singularites. First we will need a few facts.

Theorem 2.5.2. *[Thu82, Theorem 5.6] The mapping torus of a surface is hyperbolic iff the monodromy (the indentification map) is pseudo-Anosov.*

Theorem 2.5.3. *[Thu82, page 359] A hyperbolic structure on the interior of a 3-manifold has finite volume iff the boundary is a union of tori.*

This hyperbolic structure and the fact that the examples of 3-manifolds we construct only have tori as boundary components allows us to use the following theorem later.

Theorem 2.5.4 (Mostow-Prasad rigidity)**.** *If M, N are two hyperbolic manifolds of finite volume and $\pi_1(M) \cong \pi_1(N)$ then there is an isometry*

$$F : M \to N$$

which induces the isomorphism on fundamental groups.

The 3-manifolds we will consider will be hyperbolic since they are constructed as the mapping tori of pseudo-Anosov mapping classes. Their boundary will consist only of tori and so they will also be of finite volume. This allows us to use Mostow-Prasad rigidity to show that the mapping tori are homeomorphic by showing that their fundamental groups are isomorphic.

Once we have a 3-manifold we want to be able to examine the different ways the manifold fibers, which of these fibers have psuedo-Anosov monodromy, and what the dilatations are. To do all these things we consider the Thurston norm and the fibered faces of the Thurston norm ball.

Definition 2.5.5. The *Thurston norm* is

$$||\star||_T : H_2(M, \partial M, \mathbb{Z}) \to \mathbb{Z}.$$

Where given a class ϕ with a representative surface $S = \sqcup_1^n S_i$,

$$||\phi||_T = \min_S \{\sum_1^n (\max(0, -\chi(S_i)))\}.$$

The Thurston norm extends linearly to $H_2(M, \partial M, \mathbb{R})$ [Thu86, Theorem 1] and the unit sphere with respect to the Thurston norm is a convex polyhedron [Thu86, Theorem 2]. Further if an element in the cone of one of the top dimensional faces corresponds to a fibratrion then all primitive classes in that cone corresponds to fibrations of M, with pseudo-Anosov monodromy when M is hyperbolic. A face of the Thurston norm ball containing a fibration is called a *fibered face*.

When M is hyperbolic we can define a function of the fibered faces of the Thurston norm ball.

Theorem 2.5.6. *[Fri85, Proposition 8] There is a continuous, degree one homogeneous with respect to the Thurston norm, concave function $\mathcal{Y}$ tending to zero along the boundary of the fibered cone. If ϕ is fibered with pseudo-Anosov monodromy then*

$$\mathcal{Y}(\phi) = \frac{1}{\log(\lambda(\phi))}.$$

We can reformulate this to something more effective for our purposes.

Corollary 2.5.7. *There is a continuous function $\mathcal{Y}$ on the fibered face such that for each rational point ψ,*

$$\mathcal{Y}(\psi) = |\chi(\phi)|_T \log(\lambda(\phi)).$$

Here ϕ is the primitive class associated to ψ, $\chi(\phi)$ is the Euler characteristic of the fiber, and $\lambda(\phi)$ is the dilatation. Further if an element of the fibered cone corresponds to a pseudo-Anosov monodromy then all primitive classes in the cone do as well. In this case we may replace the Thurston norm with the Euler characteristic.

McMullen also shows this in [McM99, Theorem A.1]. Each rational point on a fibered face of hyperbolic 3-manifold corresponds to a fibration of this manifold with pseudo-Anosov monodromy. The continuity of the function $\mathcal{Y}$ then tells us that if we restrict to a compact subset of this face then the dilatations, $\lambda(\phi^{|\chi(S_{g,n})|})$, are bounded by a constant.

One tool for analyzing dilatations of varying fibrations on three manifolds is the Teichmuller polynomial. Another tool for anylizing different fibrations of a three manifold and the manifolds themselves is the Alexander polynomial. We include algorithms for computing both of these.

2.5.1 Computing Teichmuller Polynomials

The Teichmuller polynomial encodes the dilatations of all fibrations in a fibered face of a three manifold as the largest real root of the polynomial specialized to the particular class. The Teichmuller polynomial is defined for a fibered face of the 3-manifold. We give the computation here. more details can be found in [McM99].

Given a pseudo-Anosov mapping class $\phi : S \to S$ with invariant train track τ we consider the invariant cohomolgy $H \leq H^1(S, \mathbb{Z})$ and its corresponding cover $\tilde{S}$. The train track lifts to $\tilde{\tau}$ and ϕ lifts to $\tilde{\phi}$ inducing an action on $\tilde{\tau}$ as a $\mathbb{Z}(t_1, ..., t_n)$ module where n is the rank of H. This action acts on the space of all edge weights for τ and all vertex weights for τ. The transition matrix for the action on the edge weights is denoted P_E and the transition matrix for the action on all vertex weights is denoted P_V. Then for the fibered face F containing ϕ, the Teichmuller polynomial $\Theta_F(t_1, ..., t_n, u)$ is defined as follows:

$$\Theta_F(t_1, ..., t_n, u) = \frac{\mid Iu - P_E \mid}{\mid Iu - P_V \mid}.$$

Proposition 2.5.8. *If we are given a fibered face F of a 3-manifold, M, with Teichmuller polynomial $\Theta_F(t_1, ..., t_n, u)$ and a primitive class $(x_1, ..., x_{n+1}) \in H^1(M, \partial M, \mathbb{Z})$ then the dilatation of the pseudo-Anosov mapping class associated to $(x_1, ..., x_{n+1})$ is the largest root of $\Theta_F(t_1^{x_1}, ..., t_n^{x_n}, u^{x_{n+1}})$.*

Further we can find the *fibered cone*, the cone over the fibered face using the Teichmuller polynomial.

Theorem 2.5.9. *[McM99, Theorem 6.1] For any fibered face F of the Thurston norm ball, there exists a face D of the Teichmuller norm ball such that $\mathbb{R}^+ \cdot F = \mathbb{R}^+ \cdot D$.*

2.5.2 Computing Alexander polynomials for fibered 3-manifolds

Alexander polynomials are a classical construction which give an invariant for a knot or link complement in S^3. The Alexander polynomial is more generally an invariant of the first homology group of the maximal abelian representation for a 3-manifold.

Given a 3-manifold M and its maximal abelian cover $\tilde{M}$ we consider the homology group $H_1(\tilde{M}, \mathbb{Z})$. This homology group is not finitely presented in general but is finitely presented as a $\mathbb{Z}^n$ module for some n. We can find a presentation where $\mathbb{Z}[t]$ is a Laurent polynomial ring,

$$\mathbb{Z}[t_1, ..., t_n]^r \to \mathbb{Z}[t_1, ..., t_n]^s \to H_1(\tilde{M}, \mathbb{Z}) \to 1.$$

The gcd of the minors of the first map gives the Alexander polynomial.

Definition 2.5.10. The *Alexander polynomial* $\Delta_M(t_1, ..., t_n)$ of a 3-manifold M is the gcd of the minors of a presentation matrix for the first homology of the maximal abelian cover $\tilde{M}$.

The Alexander polynomial will be useful for finding the fibered face and determining the Thurston Norm. The Teichmuller polynomial defines the cone over the fibered face. Then in a fibered cone the Thurston and Alexander norms agree for hyperbolic manifolds. Thus the alexander polynomail defines the Thurston norm on a fibered cone and the Teichmuller polynomial determines the cone.

Theorem 2.5.11. *[McM03, Theorem 1.1] Let M be a compact, connected, orientable 3-manifold whose boundary (if any) is a union of tori. Then the Alexander and Thurston norms on $H^1(M, \mathbb{Z})$ satisfy*

$$\mid \phi \mid_A \leq \mid \phi \mid_T + \begin{cases} 0 & \text{if } b_1(M) \geq 2 \\ 1 + b_3(M) & \text{if } b_1(M) = 1 \text{ and } H^1(M, \mathbb{Z}) = \phi\mathbb{Z} \end{cases}$$

Equality holds if $\phi : \pi_1(M) \to \mathbb{Z}$ is represented by a fibration $M \to S^1$ with fibers of non-positive Euler characteristic.

CHAPTER 3

CONSTRUCTION AND APPLICATION

In this chapter we present our Generalization of Penner's construction and use it to give the asymptotic behavior for certain sequences. We consider a surface $S_{g,n,1}$ with genus g, n punctures and one boundary component. Then consider the boundary component to be partitioned into four consecutive arcs. If we consider m copies of this surface which we will call Σ_i where $0 \leq i \leq m-1$. Then we obtain the surface F_m by gluing the first arc of Σ_i to the third arc of $\Sigma_{i+1 \text{mod} m}$. A rotation map, ρ, of order m acts on the surface F_m sending Σ_i to $\Sigma_{i+1 \text{mod} m}$. The surface F_m has genus mg and if we contract the two boundary components we have $mn+2$ punctures. We also define a "connecting curve" γ between Σ_1 and Σ_2 one such curve is given in figure 3.1. The figure shows a connecting curve for F_4 with $S_{2,3,1} = \Sigma_i$. The connecting curve may also be a multicurve.

Then we choose multicurves C and D on Σ_1 such that $C \cup \gamma$ is a multicurve that intersects D minimally, and the images of $C \cup D \cup \gamma$ under ρ_m^j fill F_m where $j = 1...m$. We then choose a word $\omega \in R(C^+, D^-)$ with which we define the mapping classes $\phi_m : F_m \to F_m$ by $\phi_m := \rho_m \tau_\gamma \omega$. We now have the sequence of pairs (F_m, ϕ_m) for $m > 2$ which we call *Penner sequences.*

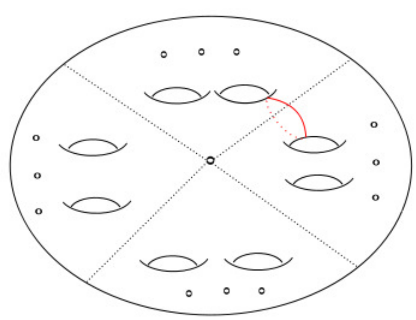

Figure 3.1: Connecting curve

Instead of constructing a train track we will find a particular invariant bigon track for Penner sequences. A bigon track is a train track with condition about the double of complementary pieces relaxed to allow for bigons. Bigon tracks are equivalent to train tracks except for the fact that the

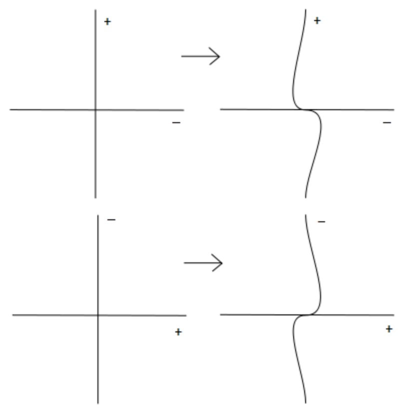

Figure 3.2: Smoothing

measures do not define unique measured laminations [PH92].

Lemma 3.0.12. *Given a Penner sequence $(\mathcal{F}, \phi_m)$ there exists invariant bigon train tracks on each surface F_m such that:*
1. The curves used to define ω and the connecting curve are carried on the bigon train track.
2. The images of the curves used to define ω and the image of the connecting curve under the map ϕ_m are carried on the bigon train track.

Proof: We can construct a bigon train track σ_m on the surface F_m so that each of the curves in $J = \cup_{i=1...m}\rho^i(C \cup D \cup \gamma)$ are caried on τ_m by locally smoothing the intersections of the curves according to the orientation the Dehn twists will be performed. The smoothing is illustrated in Figure 3.2.

If we consider a curve x that is carried by σ and perform a Dehn twist by an element $y \in J$ then the resulting curve $\tau_y(x)$ is carried by the edges carrying x and the edges that carry the element y. This is because of the choice of smoothing used to obtain the bigon train track. Since the bigon train track is symmetric with respect to the rotation map we are done.□

In terms of transverse measures we prove a similar lemma.

Lemma 3.0.13. *Consider $x, y \in J$ where $J = \cup_{i=1...m}\rho^i(C \cup D \cup \gamma)$. If μ_x and μ_y are the measures induced by the curves x and y and $\tau_x^\star$ is the map on transverse measures induced by τ_x then*

$$\tau_x^\star(\mu_y) = \mu_y + i(x, y)\mu_x$$

Proof: Collapsing is a homotopy and so the measures induced on the bigon train track are invariant with respect to homotopy. Therefore we can consider a curve as a homology class. If two curves x and y are in minimal position then performing a Dehn twist on y about x will induce a map on homology which sends y to $y + i(x, y)x$. We collapse the homology elements and obtain the desired result.□

If $\sharp C = r$ and $\sharp D = s$ are the number of components of C and D respectively then the carrying induces measures $\mu_1 ... \mu_{m(r+s+1)}$. If the curves are labled $1...m(r+s+1)$ then the measure μ_i gives an edge the weight equal to the number of disjoint arcs of the curve i that are carried by that edge. The previous Lemmas tell us that we can find a trasition matrix on these measures M_ϕ.

Lemma 3.0.14. *The matrix M_{ϕ_m} is Perron-Frobenius.*

Proof: We consider the map $(\phi_m)^m$ in which we perform Dehn twists about all the curves in $J = \cup_{i=1...m} \rho^i (C \cup D \cup \gamma)$. Since a curve x is connected to a curve y by at most $m(r+s+1)$ curves then by Lemma 3.0.3 we see that the transition matrix for $\phi_m^{m^2(r+s+1)}$ is strictly positive.$\square$

Theorem 3.0.15. *If we are given a Penner sequence $(\mathcal{F}, \phi_m)$ for $m > 2$ then ϕ_m are pseudo-Anosov with $\log(\lambda(\phi_m)) \leq \frac{P}{m}$ for some constant P.*

Proof: The mapping classes ϕ_m^m are pseudo-Anosov by Penner's semigroup criteria and so the mapping classes ϕ_m are as well. A pseudo-Anosov mapping class has an invariant train track. Lemma 3.0.12 gives an invariant bigon train track. The fact that it is a bigon track and not a train track will not be a problem as we will find a unique measure that defines the expanding lamination.

As stated in Theorem 2.4.3 the spectral radius of a Perron-Frobenius matrix is bounded by the largest column sum of the matrix. So now we would like to compute the matrix defining the action on the transverse measures. This matrix will be Perron-Frobenius by Lemma 3.0.14 and can be computed using Lemma 3.0.14.

The map ρ permutes the curves of $J = \cup_{i=1...m} \rho^i (C \cup D \cup \gamma)$ and so the induced map on the space of weights spaned by $\mu_1 ... \mu_{m(r+s+1)}$ is defined by a block permutation matrix.

$$M_\rho = \begin{pmatrix} 0 & I & 0 & . & 0 \\ 0 & 0 & I & . & 0 \\ . & . & . & . & . \\ I & 0 & 0 & . & 0 \end{pmatrix}$$

The Dehn twist about the connecting curve gives the map with transition matrix defined by:

$$M_{\tau_\gamma} = \begin{pmatrix} U & 0 & 0 & . & 0 \\ V & I & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & I \end{pmatrix}$$

Last the transition matrix for the map induced by the word ω is given below.

$$M_\omega = \begin{pmatrix} W & 0 & 0 & . & 0 \\ 0 & I & 0 & . & 0 \\ 0 & 0 & I & . & 0 \\ . & . & . & . & . \\ X & 0 & 0 & . & I \end{pmatrix}$$

Then the matrix for the map ϕ_m is given below by matrix multiplication after making the identifications $WU = Y$ and $XK = Z$.

$$M_{\phi_m} = \begin{pmatrix} 0 & Y & 0 & 0 & . & . & 0 \\ 0 & V & I & 0 & . & . & 0 \\ 0 & 0 & 0 & I & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & I \\ I & Z & 0 & 0 & . & . & 0 \end{pmatrix}$$

The matrix M is an $m \times m$ block matrix of $r+s+1 \times r+s+1$ blocks. The matrix Y depends on the word ω. The matrix V may have non-zero entries in the last column except that the last row must be zero since $\gamma \cap \rho(\gamma) = \emptyset$. The observation $V^2 = 0$ will be important later.

$$V = \begin{pmatrix} 0 & . & . & . & 0 & v_1 \\ 0 & . & . & . & 0 & v_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & v_{rs} \\ 0 & . & . & . & 0 & 0 \end{pmatrix}$$

The matrix Z will depend on ω as well but only has non-zero entries in the last row. Now we want to consider the matrix M^m.

Inductively we see that for $1 < k < m$ the matrix M^k is given by the following matrix. Here we use the fact that V^2 is the zero matrix.

$$M^k_{\phi_m} = \begin{pmatrix} 0 & 0 & 0 & . & YV & Y & 0 & 0 & . & 0 \\ . & . & . & . & 0 & V & I & 0 & . & . \\ . & . & . & . & . & 0 & 0 & I & . & 0 \\ 0 & 0 & . & . & . & . & . & 0 & . & I \\ I & Z & 0 & . & . & . & . & . & . & 0 \\ 0 & Y+ZV & Z & . & . & . & . & . & . & . \\ . & YV & Y+ZV & . & . & . & . & . & . & . \\ . & 0 & YV & . & . & . & . & . & . & . \\ . & . & 0 & . & 0 & . & . & . & . & . \\ . & . & . & . & C & 0 & . & . & . & . \\ 0 & 0 & 0 & . & Y+ZV & Z & 0 & 0 & . & 0 \end{pmatrix}$$

Then we can find the transition matrix for the mth iterate.

$$M^m = \begin{pmatrix} Y & YZ & 0 & . & . & 0 & YV \\ V & Y+ZV & Z & . & . & . & 0 \\ 0 & YV & Y+ZV & . & . & . & . \\ . & 0 & YV & . & . & . & . \\ . & . & 0 & . & . & 0 & . \\ . & . & . & . & . & Z & 0 \\ 0 & . & . & . & . & Y+ZV & Z \\ C & 0 & 0 & . & . & YV & Y+ZV \end{pmatrix}$$

Then since this matrix is Perron-Frobenius by Lemma 3.0.4 and so by Theorem 2.4.3 the spectral radius is bounded by the largest column sum. A block column sum is either equal to a column sum of $YV + Z + ZV$, $YZ + Y + ZV + YV$, or $Y + V + Z$. Therefore the dilatation of the mth iterate is bounded by a constant, say P. This tells us that

$$\log(\lambda(\phi_m)) \leq \frac{P}{m}.$$

This completes the proof.□

Theorem 3.0.16. *There is a single homeomorphism class of mapping tori for a Penner sequence.*

Proof: The proof uses a computation of the fundamental group and application of Mostow-Prasad rigidity.

The fundamental group of our surface has $m(r + s) + 1$ generators. Which are the curves $\cup_{i=1...m}\rho^i(C \cup D \cup \gamma)$ connected to a base point x_0 in Σ_1 by an arc traversing around the center puncture to the image of the base point under the rotation map and then continuing along the image of the generators for Σ_1. The last generator is a curve, c, that traverses the center puncture. The map ρ then takes the set of generators in Σ_i to the set of generators in Σ_{i+1} if $i < m$ and takes the generators for Σ_m to the generators of Σ_1 conjugated by c. The fundamental group of the mapping torus is generated by the generators of $\pi_1(F_m)$ and an extra generator t which is a section of the fibration. The relations are of the form $tg_it^{-1} = \phi^\star_m(g_i)$ where $\phi^\star_m$ is the induced map on the fundamental group. So the trick is to examine what this action on the fundamental group is. The mapping torus for ϕ_m then has fundamental group with presentation given below.

$$\langle x_i, c, t \mid tx_jt^{-1} = R_j, tx_kt^{-1} = x_{k+r+s}, tx_lt^{-1} = cx_{l-(m-1)(r+s)}lc^{-1} \rangle$$

Where $i = 1...m(r + s)$, $j = 1...2(r + s)$, $k = 2(r + s) + 1...(m - 1)(r + s)$, and $l = (m - 1)(r + s) + 1...(r + s)$. The R_j are written in terms of $x_{r+s+1}...x_{3(r+s)}$ and only depend on ω. Now we show that the mapping torus for ϕ_m is isomorphic to the mapping torus for ϕ_{m+1}.

The two presentations are

$$\langle x_i, c, t \mid tx_jt^{-1} = R_j, tx_kt^{-1} = x_{k+r+s}, tx_lt^{-1} = cx_{l-(m-1)(r+s)}lc^{-1} \rangle$$

Where $i = 1...m(r + s)$, $j = 1...2(r + s)$, $k = 2(r + s) + 1...(m - 1)(r + s)$, and $l = (m - 1)(r + s) + 1...m(r + s)$.

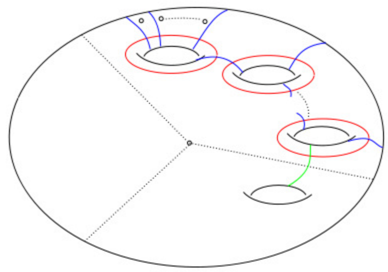

Figure 3.3: Choice for curves

And

$$\left\langle x_i, c, t \mid tx_jt^{-1} = R_j, tx_kt^{-1} = x_{k+r+s}, tx_lt^{-1} = cx_{l-(m)(r+s)}lc^{-1} \right\rangle.$$

Where $i = 1...(m+1)(r+s)$, $j = 1...2(r+s)$, $k = 2(r+s)+1...(m)(r+s)$, and $l = (m)(r+s)+1...(m+1)(r+s)$.

Now if we remove the generators x_i where $i = m(r+s)+1...(m+1)(r+s)$ with the relations $tx_{i-(r+s)}t^{-1} = x_i$ then we get the presentation:

$$\left\langle x_i, c, t \mid tx_jt^{-1} = R_j, tx_kt^{-1} = x_{k+r+s}, t^2x_lt^{-2} = cx_{l-(m-1)(r+s)}lc^{-1} \right\rangle$$

Where $i = 1...m(r+s)$, $j = 1...2(r+s)$, $k = 2(r+s)+1...(m-1)(r+s)$, and $l = (m-1)(r+s)+1...(r+s)$.

There is an isomorphism then from the presentation of the fundamental group of the mapping torus of ϕ_m to the reduced presentation of the fundamental group of the mapping torus of ϕ_{m+1} defined by $c \to ct^{-1}$ and the identity on all other generators. Since the mapping tori are all hyperbolic manifolds and their fundamental groups are isomorphic they are homeomorphic by Mostow-Prasad rigidity.□

Next we use this theorem to show the asymptotic behavior of the minimal dilitation for certain rational rays in the gn-plane.

Theorem 3.0.17. *Given a rational ray in the gn-plane emanating from the point $(0,0)$, $(1,0)$ or $(2,0)$ which contains the points (n_i, g_i) we have:*

$$\log(\delta_{g_i,n_i}) \asymp \frac{1}{|\chi(S_{g_i,n_i})|}.$$

Proof: With Penner's lower bound we only need the upper bound to prove the asymptotic behavior. Suppose a ray has slope $\frac{p}{q}$ with $(p,q) = 1$ then let $S_{g,n,1}$ have $g = p$, and $n = q$. The proof is done for rays emanating from $(2,0)$. Further if we choose our curves as in Figure 3.3, which is shown with a chosen connecting curve as well, then we can find a train track for ϕ_m given in Figure 3.4.

From this we can see by Lemma 2.3.1 that the two fixed punctures are not 1-pronged. Filling in one of the punctures then gives a sequence of mapping classes with the same dilatation and one fewer

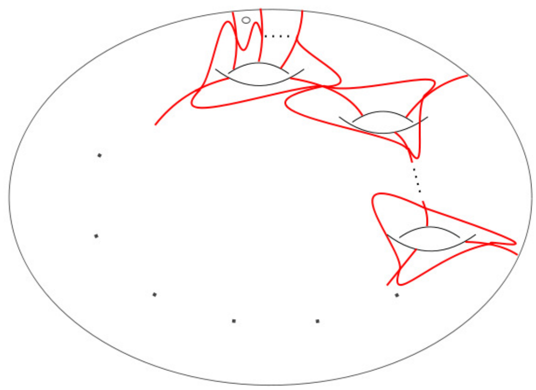

Figure 3.4: Train track

puncture, the sequences for rays passing through $(1,0)$. Filling in both fixed punctures we obtain sequences of mapping classes with the same dilatation and two fewer punctures, the sequences for the rays passing through the origin.□

CHAPTER 4

EXAMPLES

In this chapter we would like to consider a particular set of examples illustrating our main theorm and examine their mapping tori in terms of their fibered faces.

We consider the examples generated, as in chapter 3, by $S_{1,0,1}$ the surface of genus 1 with, 0 punctures and 1 boundary component and the word $\omega = \tau_a^{-r}\tau_b^s$ for some positive integers (r, s). The curves a, b, and the connecting curve c are given in Figure 4.1.

Penner's examples correspond to $(r, s) = (1, 1)$. Since by our main theorem we know that given (r, s) all the mapping tori are homeomorphic we only need to consider a low genus example to compute the Alexander and Teichmuller polynomials which will give us a better idea of what the fibered faces look like.

4.1 Alexander and Teichmuller polynomials

Now we know that a sequence of mapping classes defined by the word ω has a single mapping torus $M_{r,s}$. We can ask what other mapping classes have the same mapping torus? We start by computing the Alexander and Teichmuller polynomials. We may consider the monodromies to be on genus 4 surfaces with 2 points removed, corresponding to the axis of rotation. One may compute the fundamental group of the mapping torus using the fundamental group of the surface and an extra generator given by a section of the fibration containing the base point.

The fundamental group of the surface is a free group generated by the curves given by the a, b, c and their images under the rotation map along with a path from the basepoint around the center puncture. The generators are depicted in Figure 4.2. Now we need to understand how each of the Dehn twists and the rotation act on these generators.

We consider the action of individual parts of the map.

- $\tau_{a_1}^{-r}(\beta_1) = \beta_1\alpha_1^r$
- $\tau_{a_1}^{-r}(\gamma_1) = \gamma_1\alpha_1^r$
- $\tau_{a_1}^{-r}(\gamma_g) = c\alpha_1^{-r}c^{-1}\gamma_g$
- $\tau_{b_1}^{s}(\alpha_1) = \beta_1^s\alpha_1$

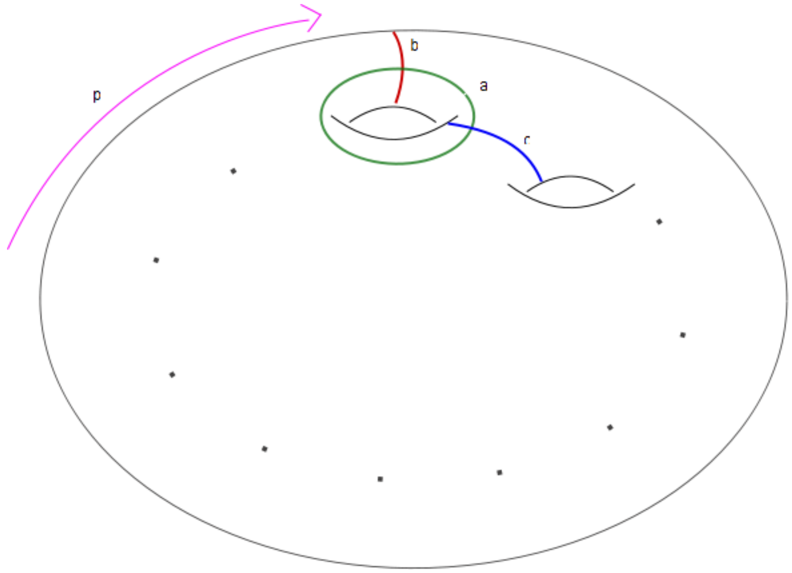

Figure 4.1: Curves

- $\tau_{c_1}(\alpha_1) = \gamma_1^t \alpha_1$
- $\tau_{c_1}(\alpha_2) = \gamma_1 \alpha_2$
- $\tau_{c_1}(\beta_1) = \gamma_1 \beta_1 \gamma_1^{-1}$

All the other generators for the fundamental group of the surface are left fixed by these Dehn twists. The action of the rotation ρ is given by $\rho(\alpha_i) = \alpha_{i+1}$ for $1 \leq i < g$, $\rho(\alpha_g) = c\alpha_1 c^{-1}$, likewise for β_i and γ_i, last $\rho(c) = c$. From this we can find the action of $\phi_{g,(r,s)}$ on the fundamental group of the surface. The fundamental group of the mapping torus $M_{\phi_{(r,s)}}$ has a presentation with generators α_i, β_i, γ_i, c, and f where f is the suspension of the base point and relations $fxf^{-1} = \phi_{g,(r,s)}(x)$ where x is a generator for the fundamental group of the surface. From this we get a presentation for the fundamental group of the mapping torus $M_{\phi_{(r,s)}}$.

$$\pi_1(M_{(r,s)}, x_0) = \langle \alpha_i, \beta_i, \gamma_i, c, f \mid f\alpha_1 f^{-1} = (\gamma_2\beta_2\gamma_2^{-1}(\gamma_2^t\alpha_2)^r)^s \gamma_2^t \alpha_2$$

$$f\beta_1 f^{-1} = \gamma_2\beta_2\gamma_2^{-1}(\gamma_2\alpha_2)^r$$

$$f\gamma_1 f^{-1} = \gamma_2(\gamma_2\alpha_2)^r$$

$$f\alpha_2 f^{-1} = \gamma_2\alpha_3$$

$$f\gamma_g f^{-1} = c(\gamma_2\alpha_2)^{-r}\gamma_1 c^{-1}$$

$$f\alpha_i f^{-1} = \alpha_{i+1} \text{ for } 2 < i < g$$

$$f\alpha_g f^{-1} = c\alpha_1 c^{-1}$$

$$f\beta_i f^{-1} = \beta_{i+1} \text{ for } 1 < i < g$$

$$f\beta_g f^{-1} = c\beta_1 c^{-1}$$

$$f\gamma_i f^{-1} = \gamma_{i+1} \text{ for } 1 < i < g$$

$$fc = cf\rangle$$

From the fundamental group one sees that the maximal abelian cover has deck transformation group $\mathbb{Z}^2$. The invariant cohomology is the map that takes the generator c to 1. We then cut our genus 4 surface on an arc connecting the 2 punctures between the first and last handles, call the cut surface S'. This is how we will build $\tilde{S}$ for all Teichmuller and Alexander polynomials. For the Alexander polynomials we will need the action on the homology of $\tilde{S}$ and for the Teichmuller polynomial we will need the action on the train track $\tilde{\tau}$. We start with the Alexander polynomial.

The surface $\tilde{S} = \cup_{i\in\mathbb{Z}} S'_i$ with S'_i attached to S'_{i+1} by an arc on the boundary. From the construction we can see that the surface S' has homotopy type of the wedge of 8 circles and the gluing is homotopy equivalent to a wedge product. Thus $H_1(\tilde{S})$ is finitely generated by a lift of the generators for S' as a $\mathbb{Z}[y]$ module and is free. We choose generators corresponding to the a_i and b_i curves used to define the monodromy. Then we can compute the action on the homology of $\tilde{S}$ to be the following matrix.

$$M = \begin{pmatrix} 0 & A & B & 0 \\ 0 & C & D & 0 \\ 0 & 0 & 0 & I \\ Iy & 0 & 0 & 0 \end{pmatrix}$$

The matrix M is in 2×2 block form.

$$A = \begin{pmatrix} 1+r & -r \\ -s-(sr+1) & sr+1 \end{pmatrix} B = \begin{pmatrix} -r & 0 \\ (sr+1) & 0 \end{pmatrix} C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} D = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

A block I is the 2×2 identity with y being the covering transformation corresponding to the invariant cohomology.

Now $H_1(\tilde{M_{r,s}}, \mathbb{Z})$ is generated by a_i and b_i as a $\mathbb{Z}[x, y]$ module. However, there are relations corresponding to the fibration ϕ. Let $\tilde{\phi}$ be the action on the homology of $\tilde{S}$. Then $(\gamma, r) = (\tilde{\phi}(\gamma), r-1)$ and so the presentation matrix is given to us as $Ix - M$. This matrix is square and so the determinant is the Alexander polynomial.

$$\Delta_{r,s}(x, y) = x^8 + rx^5y - (rs + 2r + 2)x^4y + rx^3y + y^2$$

For the computation of the Teichmuller polynomial we consider the same lift of our fiber surface $\tilde{S}$. We also lift the train track and consider the action of $\tilde{f}$ on the space of weights for the train track. For our examples the action on this lifted track is the same as the original except that the last handle is sent to a translate of the first handle instead of just the first handel. Then the action is given by the following matrix.

$$M = \begin{pmatrix} 0 & A & 0 & 0 & . & 0 \\ . & B & I & 0 & . & . \\ . & 0 & 0 & I & . & . \\ . & . & . & 0 & . & 0 \\ 0 & 0 & . & . & . & I \\ Iy & Cy & 0 & 0 & . & 0 \end{pmatrix}$$

The matrix M is in 3×3 block form where I is the identity matrix.

$$A = \begin{pmatrix} rs+1 & s & rs+1 \\ r & 1 & r \\ r & 0 & r+1 \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r & 0 & r \end{pmatrix}$$

Then we can compute the Teichmuller polynomial $\Theta_F(x, y)$ to be:

$$\Theta_F(x, y) = (x^8 - rx^5y - (rs + 2r + 2)x^4y - rx^3y + y^2)(-y + x^4)$$

Remark: In our examples the space of admissible measures is exactly the measures induced by the curves we use to define the mapping classes. The polynomails we get are the Teichmuller Polynomials for the fibered faces of the 3-manifolds in question since the space is free [McM99, Remark 2 to theorem 3.6].

One question that is imediately raised is does this similarity in coefficents tell us anything about the manifolds in question? Together these polynomials give us a way to analyze the fibered cone the given fibration lies in. The Teichmuller polynomial determines the cone [McM99, Theorem 6.1] and the Alexander polynomial determines the Thurston norm on the cone [McM03, Theorem 1.1].

Lemma 4.1.1. *The Thurston norm on the fibered face is* $| (x, y) |_T = 8x - 2y$ *and all classes have two boundary components.*

Proof: The Thurston and Alexander norms agree on the fibered face and so looking at the Alexander polynomial we see that $| (a, b) |_T = 8a - 2b$. As for the boundary components of a class (a, b) we consider the intersection of the fiber with a torus homotopic to the boundary. A curve around the top center puncture is a meridian for one boundary component whoose longitude is the meridian for the other. Therefore a class (a, b) intersects the torus in a (a, b) curve. Since a and b are relatively prime there is only one connected component to this curve. Likewise for the other boundary component.□

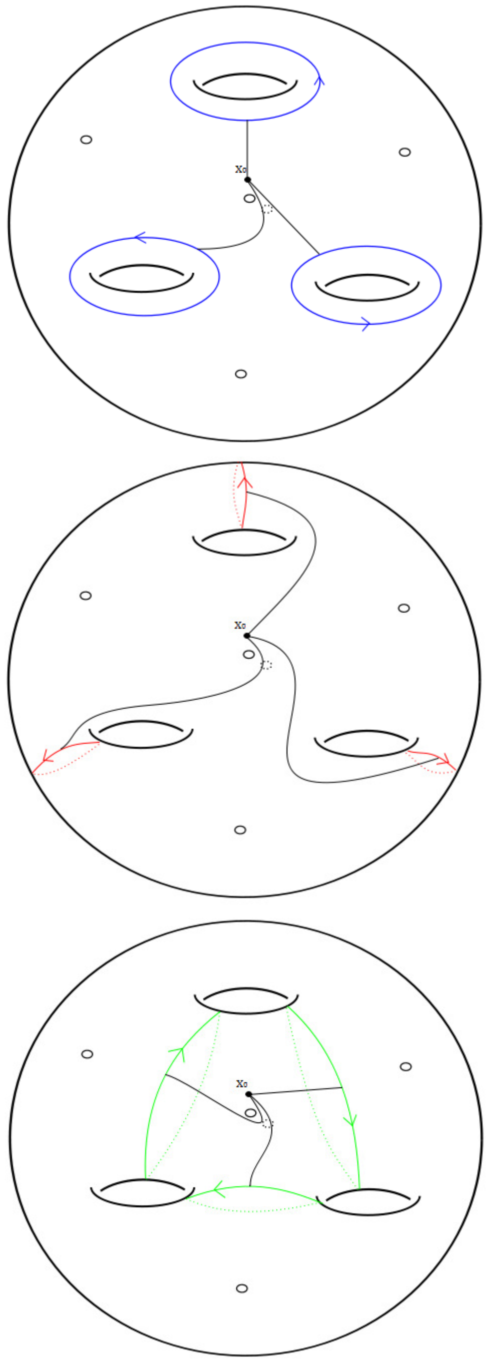

Figure 4.2: Fundamental group generators

APPENDIX A

OTHER RAYS AND THE MAGIC MANIFOLD

In this chapter we give some partial results about asymptotic behavior on other rational rays. The method presented in this thesis for finding asymptotic behavior on rays through the origin and the points $(1, 0)$ and $(2, 0)$ can be viewed as finding a class of examples for a vertical ray through a point in the gn-plane and adjusting these examples to get examples for the other rational rays through that point. In this chapter we present this methodology to give an approach to finding rational rays through other points $(n, 0)$. In the first section we present the method. In the second section we find examples for arithmetic sequences on vertical rays in the gn-plane coming from the magic manifold which give the same asymptotic bound as for closed surfaces. This gives hope that it is possible to use our method to show that the asymptotic behavior for all rational rays through points $(n, 0)$ is the same as for closed surfaces.

A.1 A method for finding asymptotic behavior

In this section we create a strategy for examining other rational rays. Our strategy for proving the asymptotic behavior on rational rays through the points $(0, 0)$, $(1, 0)$, and $(2, 0)$ was to give a Penner sequence on the rational ray. Another way to do this is to consider the set of fixed and periodic points of a mapping class.

Consider the sequence of surfaces with fixed punctures and increasing genus. Assume there is a map $\rho : S_{g,n} \to S_{g,n}$ which is periodic of order g. Take a set of curves c_j with $j = 1 \dots k$ such that the set

$$\{\rho^i(x) | x = c_j \; i = 1 \dots g \; j = 1 \dots k\}$$

fills the surface and so that we can assign positive or negative Dehn twists, $\tau_{c_j}^{\pm 1}$, about them in a way that satisfies the conditions for Penner's semigroup criteria. We then consider the maps

$$\phi_g = \rho \circ \Pi_{j=1}^{k} \tau_{c_j}^{\pm 1}.$$

This gives a Penner sequence. The dilatation of the gth power of the map can then be bounded from above by a constant. This bound gives us the upper bound for the asymptotic behavior while Penner's lower bound gives us the lower bound.

We then show that if we iterate a single Dehn twist say τ_{c_1} we add cyclic orbits by imagining the iterated Dehn twist to be a number of parallel copies of the curve. Since the curves fill the surface another curve will cut through these parallel copies leaving discs which are permuted around the surface with order g. Then there will be an order g cyclic point in each disc bounded by two of the copies of the iterated curve and some other curve. Using this method of finding cyclic points we will be able to give upper bounds for all rays through a given point $(n, 0)$ given the upper bound for the vertical ray.

Criteria A.1.1 (Asymptotic criteria)**.** *:*

- ⋆ *A periodic map ρ of order g for each surface $S_{g,n}$ (n is fixed).*

- ⋆ *A set of curves γ_i, $i = 1..m$ for each surface such that the map*

$$\phi^g = \Pi_{i=1}^m(\tau_{\gamma_i}^{s_i})\Pi_{i=1}^m(\tau_{\rho(\gamma_i)}^{t_i})...\Pi_{i=1}^m(\tau_{\rho^{g-1}(\gamma_i)}^{u_i})$$

where $s_i, t_i, u_i = \pm 1$, is pseudo-Anosov with dilataion bounded by a constant, C, for all g.

- ⋆ *A train track σ which is invariant under the map*

$$\phi = \rho \circ \Pi_{i=1}^m(\tau_{\gamma_i}^{\pm 1}).$$

Each curve $\rho^j(\gamma_i)$ for $j = 1 \dots g$ is carried by σ and each $\phi^j(\gamma_i)$ is carried σ as well.

The largest column sum of each transition matrix for ϕ^g is bounded by C and so

$$\log(\lambda(\phi)) \leq \frac{C}{\mid \chi_{g,n} \mid}.$$

This gives the desired upper bound for the vertical ray through $(n, 0)$. Now we define the new mapping class:

$$\phi' = \rho \circ \tau_{\gamma_1}^{\pm q} \circ \Pi_{i=2}^m(\tau_{\gamma_i}^{\pm 1}).$$

Proposition A.1.2. *If ϕ^g satisfies Penner's criteria for being pseudo-Anosov then so does ϕ'^g.*

Lemma A.1.3. *The largest eigenvector of the transition matrix for ϕ'^g is bounded by qC.*

Proof: To find the unique largest eigenvector we only need to consider the cone of measures associated to the γ_i since they are carried along with their images by the train track. Each entry of the transition matrix will increase by at most $q - 1$ since where as before there was a single Dehn twist adding a single strand for each intersection of γ_1 with a curve, now there are q strands added for each intersection with a curve. If an entry of the transition matrix for ϕ^g is e_j then the corresponding entry in the transition matrix for ϕ'^g is less than qe_j. Therefore a column sum in the transition matrix for ϕ'^g is less than $q\sum_j e_j \leq qC$.□

Lemma A.1.4. *Increasing the power of γ_1 by 1 yields a map ψ with an extra g-cyclic orbit.*

Proof: There is an annulus bounded by two curves both homotopic to γ_1. The union of all the curves fills the surface so there is a curve along which we will perform a Dehn twist that hits γ_1 efficiently and cuts the annulus into a disc. This disc, D, is invariant under each τ_{γ_i} and has a g-cyclic orbit under ρ. Therefore $\psi^g(D) = D$ and D has a fixed point.□

By Lemma 2.3.4 we can puncture all $q-1$ of the g-cyclic orbits and get a surface with $(q-1)g$ extra punctures and induced mapping class with the same dilatation as ψ'. This gives the bounding examples for rays of slope $\frac{1}{q-1}$ through the point $(n, 0)$

Theorem A.1.5. *If there exist mapping classes for a vertical ray passing through the point $(n, 0)$ satisfying the asmptotic criteria then we have*

$$\log(\delta_{g,n}) \asymp \frac{1}{\mid \chi_{g,n} \mid}$$

for all rational rays through the point $(n, 0)$

Proof: If there is a set of mapping classes for a vertical ray through $(n, 0)$ satisfying the asymptotic criteria then combining the upper bound with Penner's lower bound we have the asymptotic behavior for the vertical ray. Then if we want to give the asymptotic behavior for another rational ray we only need to give the upper bound $\log(\delta_{g,n}) \leq \frac{Q}{g}$ where Q is some constant.

The mapping classes for a vertical ray will have the form

$$\phi_g = \rho \circ \Pi_{i=1}^{m} \tau_{\gamma_i}^{\pm 1}.$$

We also define the mapping class

$$\phi'_g = \rho \circ \tau_{\gamma_1}^{q+1} \circ \Pi_{i=2}^{m} \tau_{\gamma_i}^{\pm 1}$$

If we want a sequence of mapping classes for a ray with slope $\frac{p}{q}$ then we consider the following mapping classes on the surfaces with fixed number of punctures n and genus $g = \frac{p}{q}(x-n)$ for all x such that g is an integer.

$$\psi_{\frac{p}{q}(x-n)} = \rho^p \phi'_{\frac{p}{q}(x-n)} \phi^{p-1}_{\frac{p}{q}(x-n)} \rho^{-p}$$

This mapping class has q more $\frac{g}{p}$ cyclic orbits than ϕ and so puncturing these orbits gives the surfaces $S_{g,b}$ satisfying the equation $g = \frac{p}{q}(b-n)$ with induced mapping class $\tilde{\psi}_{\frac{p}{q}(x-n)}$.

Further the mapping class $\psi_{\frac{p}{q}(x-n)}$ is a conjugate of $\phi'_{\frac{p}{q}(x-n)} \phi^{p-1}_{\frac{p}{q}(x-n)}$. The mapping class $(\phi'_{\frac{p}{q}(x-n)} \phi^{p-1}_{\frac{p}{q}(x-n)})^{\frac{x-n}{q}}$ is pseudo-Anosov by Penner's semigroup criteria and so the mapping class $\psi_{\frac{p}{q}(x-n)}$ is as well. Each of ϕ and ϕ' has an associated transition matrix on the train track σ with the logarithm of the spectral radius bounded by $\frac{P}{\frac{p}{q}(x-n)}$ and $\frac{P'}{\frac{p}{q}(x-n)}$ respectively since $\frac{p}{q}(x-n)$ is the genus of the surface. Therefore, $\log(\lambda(\psi_{\frac{p}{q}(x-n)})) \leq (\frac{\max P, P'}{g})^p \leq \frac{Q}{g}$ where $Q = \max(P, P')^p$. This gives us the required upper bound for a rational ray through the point $(n, 0)$.□

A.2 Partial vertical rays

We would like to find examples for vertical rays that give asymptotic bounds so that we can apply the results from the previous section. We will find examples of arithmetic sequences on vertical rays but they are not nesecarilly of the form perscribed by the asymptotic criteria.

Theorem A.2.1. *There exists an infinite sequence of examples ϕ_i for each vertical line in the $n \times g$ plane with $n \geq 4$ such that $\log(\lambda(\phi_i)) \leq \frac{C}{|\chi(\phi_i)|}$ for some constant C which is not dependent on the vertical line in question.*

Theorem A.2.2. *Fixing n, there is an accumulation point for the sequence $\mid \chi_{g,n} \mid \log(\delta_{g,n})$ which is less than $2\log(2+\sqrt{3})$*

The second theorem is related to the following question.

Question A.2.3. *Does $\lim\limits_{|\chi_{g,n}|\to\infty} \mid \chi_{g,n} \mid \log(\delta_{g,n})$ exist and if so what is the limit?*

This question was asked in [McM99] and in [Far06, Question 7.1]. For closed surfaces Hironaka gives a bound for $\limsup\limits_{g\to\infty}(\delta_{g,0})^g$.

Proposition A.2.4. *[Hir10] We have the inequality:*

$$(\limsup_{g\to\infty} \delta_g)^g \leq \frac{3+\sqrt{5}}{2}.$$

We consider classes found in the magic manifold to obtain our results. This particular manifold was studied by Kin and Takasawa in [KT09] and [KT10]. The magic manifold is the alternating 3 chain link in S^3 which has 3 dimensional 2nd homology. We notice that we may choose as generators the twice punctured discs which bound a link component and are punctured by the other components once each. Using these generators Kin and Takasawa give equations for the Thurston norm and number of boundaries for each class in the fibered face defined by the points $(1,1,1)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,-1)$.

Theorem A.2.5. *[KT09] For classes (x,y,z) in the fibered cone defined above we have the following.*

$$\mid (x,y,z) \mid_T = x+y-z$$

and

$$\sharp(\partial(x,y,z)) = \gcd(x,y+z) + \gcd(y,x+z) + \gcd(z,x+y).$$

Further, because of the symmetry of the magic manifold, each other face of the Thurston norm ball is fibered and is similar to this face.

The previous lemma tells us how to compute the number of boundary components and Euler characteristic of classes in the fibered face. The next few lemmas will allow us to find aritmetic sequences for a given number of boundary components.

Lemma A.2.6. *If a class, (x, y, z), does not lie in the planes $x - y = \pm z$ or $z = 0$ then the surfaces corresponding to the classes $(x + m, y + m, z)$ have a periodic number of boundary components as m increases.*

Proof: The number of boundary components is given by

$$\gcd(x + m, y + z + m) + \gcd(y + m, x + z + m) + \gcd(z, x + y + 2m).$$

This can be adjusted by the Euclidean algorithm to

$$\gcd(x + m, y + z - x) + \gcd(y + m, x + z - y) + \gcd(z, x + y + 2m).$$

The first term is periodic iff $y + z - x \neq 0$, therefore the first term is periodic iff the class (x, y, z) does not lie in the plane $x - y = z$. The second term is then periodic iff (x, y, z) does not lie in the plane $x - y = -z$ and the third iff (x, y, z) does not lie in the plane $z = 0$. If all three are periodic then the sum is.□

Lemma A.2.7. *If the number of boundary components are fixed then the genus is increasing on the ray through (x, y, z) in the direction $(1, 1, 0)$.*

Proof: The ray $(x + m, y + m, z)$ has Thurston norm $x + y - z + 2m$. If the number of boundary components is fixed then the genus must be increasing with m.□

Now if we have a class not on the 2 planes that we mentioned with a particular number of boundary components we can create an infinite number of examples with the same number of boundary components. Therefore we will need a class for each n.

Lemma A.2.8. *The surface corresponding to the class $(2n - 3, n - 1, 1)$ has $n + 1$ boundary components and genus $g = n - 2$ for $3 \nmid n$, $n > 3$.*

Proof: The number of boundaries is given by

$$\gcd(2n - 3, n) + \gcd(n - 1, 2n - 2) + \gcd(1, 3n - 4).$$

Adjusting by Euclidean algorithim we get

$$\gcd(3, n) + \gcd(n - 1, n - 1) + \gcd(1, 0).$$

Then the number of boundaries is $n + 1$. Using the Euler characteristic we have

$$3n - 5 = 2g - 2 + (n + 1),$$

which gives $g = n - 2$.□

Lemma A.2.9. *The class $(6n - 2, 3n - 2, 1)$ has $3n + 1$ boundaries and genus $g = 3n - 2$ for $n \geq 1$.*

Proof: The number of boundary components is given by

$$\gcd(6n - 2, 3n - 1) + \gcd(3n - 2, 6n - 1) + \gcd(1, 9n - 4).$$

This simplifies to $3n + \gcd(3n-2, 3n+1)$. Here we are taking the gcd of two numbers whose difference is three neither of which is divisible by three. Therefore we have $3n+1$ boundaries. Next the Euler characteristic gives us

$$9n - 5 = 2g - 2 + 3n + 1,$$

which simplifies to $g = 3n - 2$.□

We use the above to show Theorem A.2.1.

Proof of Theorem A.2.1: Lemma A.2.6 and Lemma A.2.7 give us a single class with a particular number of boundary components. Each of these classes is off the planes $x - y = \pm z$ and $z = 0$ so we can consider the rays $(x+m, y+m, z)$ and Lemmas A.2.4 and A.2.5 apply which tell us that there are infinitely many classes in this ray with the same number of boundary components as the original and increasing genus. Furthermore the sequence of points in Lemmas A.2.6 and A.2.7 are parallel to the ray through $(2,1,0)$ the set of all these points are contained in a closed subset of the open fibered face. This tells us that if the classes in Lemmas A.2.6 and A.2.7 are denoted ϕ_n where n is the number of boundary components then $\mathcal{Y}(\phi_n)$ is bounded by some number C. If we consider $\phi_n = (x,y,z)$ for a particular n when the points $(x+m, y+m, z)$ are projected to the face they are contained in the closed line segment with endpoints the projection of (x,y,z) and $(\frac{1}{2}, \frac{1}{2}, 0)$ to the face. Then if we take the closed subset that contains all the ϕ_n and cone it over the point $(\frac{1}{2}, \frac{1}{2}, 0)$ then we have a closed subset of the fibered face that contains classes for each sequence. Therefore $\mathcal{Y}((x+m, y+m, z)) \leq C$ for all m.□

Proof of Theorem A.2.2: The sequence of classes for each n is approaching $(\frac{1}{2}, \frac{1}{2}, 0)$. $\mathcal{Y}(\frac{1}{2}, \frac{1}{2}, 0) = 2\log(2 + \sqrt(3))$.□

This gives us a partial result for the asymptotic behavior for vertical rays but we would need to know what the maps are in order to exploit our method to find asymptotic behavior for arithmetic sequences on the rational rays through these points.

BIBLIOGRAPHY

[AD10] John Aaber and Nathan Dunfield. Closed surface bundles of least volume. *Algebraic and geometric topology*, 10:2315–2342, 2010.

[Ago10] Ian Agol. Ideal triangulation of pseudo-anosov mapping tori. *arXiv:1008.1606*, 2010.

[bDKM11] A. Fathi; F. Laudenbach; V. Poenaru; Translated by Djun Kim and Dan Margalit. *Thurston's Work on Surfaces*. Princeton University Press, 2011.

[CH08] Jin-Hwan Cho and Ji-Young Ham. The minimal dilatation of a genus-two surface. *Experiment. Math*, 17:257–267, 2008.

[EKT09] S. Kojima E. Kin and M. Takasawa. Entropy versus volume for pseudo-anosovs. *Experimental Mathematics*, 18:397–407, 2009.

[EL11] Jean-Luc Thiffeault Erwan Lanneau. On the minimal dilatation of pseudo-anosov homeomorphisms on surfaces of small genus. *Annales de l'Institut Fourier*, 61 (1):105–144, 2011.

[Far06] Benson Farb. Some problems on mapping class groups and moduli space. *Proceedings of symposia in pure mathematics*, 74, 2006.

[Fri85] David Fried. Growth rate of surface homeomorphisms and flow equivalence. *Ergodic Theroy and Dynamical Systems*, 5(4):539–563, 1985.

[Ham06] J. Ham. *The Minimum Dilatation of 4 and 5 Braids*. PhD thesis, University of California, Santa Barbara, 2006.

[Hir10] Eriko Hironaka. Small dilatation pseudo-anosov mapping classes coming from the simplest hyperbolic braid. 10:2041–2060, 2010.

[HK06] E. Hironaka and E. Kin. A family of pseudo-Anosov braids with small dilatation. *Algebraic and Geometric Topology*, 6:699–738, 2006.

[HS07] J. Ham and W. T. Song. The minimum dilatation of pseudo-anosov 5-braids. *Experiment. Math*, 16:224229, 2007.

[Iva90] N. V. Ivanov. Stretching factors of pseudo-anosov homeomorphisms. *Journal of Mathematical Sciences*, 52:2819–2822, 1990. 10.1007/BF01099245.

[KT09] Eiko Kin and Mitsuhiko Takasawa. Pseudo-anosov braids with small entropy and the magic 3-manifold. *Communications in Analysis and Geometry*, 2009.

[KT10] Eiko Kin and Mitsuhiko Takasawa. Pseudo-anosovs on closed surfaces having small entropy and the whitehead sister link exterior. *(Preprint)arXiv:0909.4517*, 2010.

[LM09] Benson Farb; Christopher J. Leininger; and Dan Margalit. Small dilatation psuedo-anosovs and 3-manifolds. *(Preprint)arXiv:0905.0219v1*, 2009.

[McM99] C.T. McMullen. Polynomial invariants for fibered 3-manifolds and teichmuller geodesics for foliations. *Annales Scientifiques de l'cole Normale Suprieure*, 1999.

[McM03] Curtis McMullen. The alexander polynomial of a 3-manifold and the thurston norm on cohomology. *Ann. Sci. cole Norm. Sup*, 2003.

[Pen88] R.C. Penner. A construction of pseudo-Anosov homeomorphisms. *Transactions of the American Mathematical Society*, 310(1):179–197, 1988.

[Pen91] R.C. Penner. Bounds on least dilatations. *Proceedings of the American Mathematical Society*, 113(2):443–450, 1991.

[PH92] R.C. Penner and J.L. Harer. *Combinatorics of Train Tracks.* Princeton University Press, Princeton, NJ, 1992.

[PR87] A. Papadopoulos and R.C.Penner. A characterization of pseudo-Anosov foliations. *Pacific Journal of Mathematics*, 130(2), 1987.

[Thu82] William P. Thurston. Three dimensional manifolds, kleinian groups and hyperbolic geometry. *Bulletin of the American Mathematical Society*, 6(3), 1982.

[Thu86] William Thurston. A norm for the homology of 3-manifolds. *Memoirs of the American Mathematical Society*, 59(339), 1986.

[Tsa09] Chai-Yen Tsai. The asymptotic behavior of least pseudo-Anosov dilatations. *Geometry and Topology*, 2009.

[Yoc81] Pierre Arnoux; Jean-Christophe Yoccoz. Construction de diffomorphismes pseudo-anosov. *C. R. Acad. Sci. Paris Sr. I Math.*, 292(1):7578, 1981.

BIOGRAPHICAL SKETCH

Born in Columbia, Maryland on August 8, 1984 Aaron attended Howard County Community College and Salisbury University graduating with a Bachelors of Science in the spring of 2006. He moved to Tallahassee in the summer of 2006 to attend Florida State University for graduate school and successfully defended in the Fall of 2011.